ME 401
Advanced Kinematics

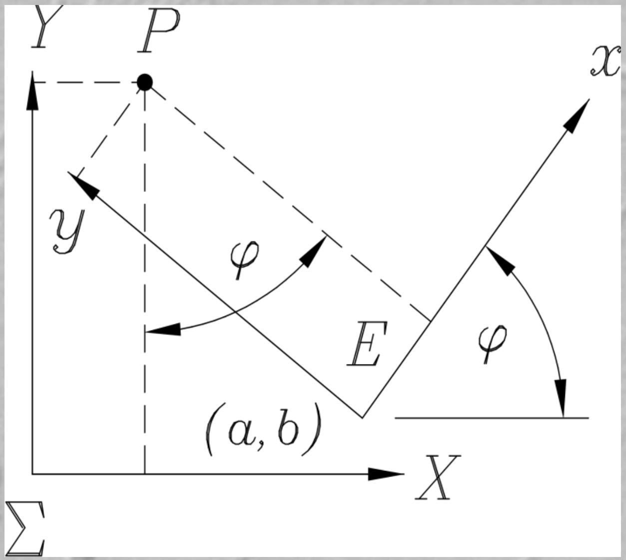
UNIT 3

Kinematics synthesis of advance topics

Department of Mechanical Engineering

Planar Kinematic Mapping

- Three parameters, a, b and ϕ describe a planar displacement of E with respect to Σ.
- The coordinates of a point in E can be mapped to those of Σ in terms of a, b and ϕ:



$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  - (x:y:z): homogeneous coordinates of a point in E.
  - (X:Y:Z): homogeneous coordinates of the same point in Σ.
  - (a,b): Cartesian coordinates of O_E in Σ.
  - ϕ: rotation angle from X- to x-axis, positive sense CCW.

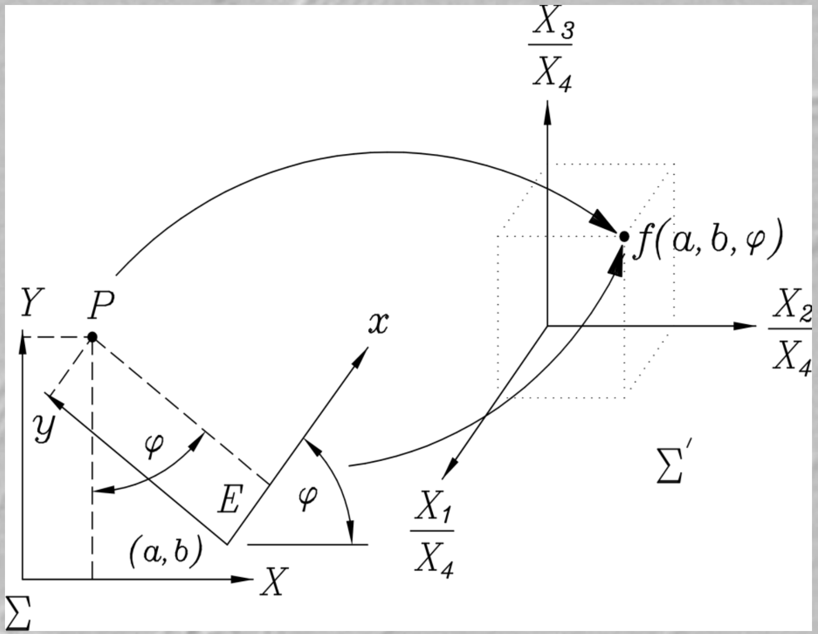
Kinematic Mapping

- The mapping takes distinct poles to distinct points in a 3-D projective image space. It is defined by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} a\sin(\varphi/2) - b\cos(\varphi/2) \\ a\cos(\varphi/2) + b\sin(\varphi/2) \\ 2\sin(\varphi/2) \\ 2\cos(\varphi/2) \end{bmatrix}$$

- Dividing by X_4 normalizes the coordinates:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(a\tan(\varphi/2) - b\right) \\ \frac{1}{2}\left(a + b\tan(\varphi/2)\right) \\ \tan(\varphi/2) \\ 1 \end{bmatrix}$$



- The inverse mapping is:

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2) \end{aligned}$$

Kinematic Mapping

- Using half-angle substitutions and these above relations the basic Euclidean group of planar displacements can be written in terms of the image points

$$\lambda\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- The inverse transformation yields

$$\mu\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & 2X_3X_4 & 2(X_1X_3 - X_2X_4) \\ -2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 + X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

- λ and μ being non-zero scaling factors arising from the use of homogeneous coordinates.

Constraint Manifold Equation

- Consider the motion of a fixed point in *E* constrained to move on a fixed circle in Σ, with radius *r*, centred on the homegeneous coordinates ($X_C : Y_C : Z$) and having the equation

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0,$$

where

$$K_0 = \text{arbitrary homogenising constant.}$$

  - If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

  - If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

PR-Dyad Line Coordinates

- For *PR*-dyads the K_i line coordinates are generated by expanding the determinant created from the coordinates of a known point on the line, and the known direction of the line, both fixed relative to Σ:

$$\begin{vmatrix} X & Y & Z \\ F_{X/\Sigma} & F_{Y/\Sigma} & 1 \\ \cos\xi_\Sigma & \sin\xi_\Sigma & 0 \end{vmatrix}$$

where

$$\begin{aligned} X, Y, Z &= \text{homogenious coordinates of points on the line,} \\ F_{X/\Sigma}, F_{Y/\Sigma} &= \text{coordinates of fixed point on the line in } \Sigma, \\ \xi_\Sigma &= \text{angle of the line relative to } \Sigma. \end{aligned}$$

giving

$$\left[K_1 : K_2 : K_3\right] = \left[-\frac{1}{2}\sin\xi_\Sigma : \frac{1}{2}\cos\xi_\Sigma : F_{X/\Sigma}\sin\xi_\Sigma - F_{Y/\Sigma}\cos\xi_\Sigma\right].$$

RP-Dyad Line Coordinates

- For *RP*-dyads the K_i line coordinates are generated by expanding the determinant created from the coordinates of a known point on the line, and the known direction of the line, both fixed relative to *E*:

$$\begin{vmatrix} x & y & z \\ M_{x/E} & M_{y/E} & 1 \\ \cos\xi_E & \sin\xi_E & 0 \end{vmatrix}$$

where

$$\begin{aligned} x, y, z &= \text{homogenious coordinates of points on the line,} \\ M_{x/E}, M_{y/E} &= \text{coordinates of a fixed point on the line in } E, \\ \xi_E &= \text{angle of the line relative to } E. \end{aligned}$$

giving

$$[K_1 : K_2 : K_3] = \left[-\frac{1}{2}\sin\xi_E : \frac{1}{2}\cos\xi_E : M_{x/E}\sin\xi_E - M_{y/E}\cos\xi_E\right].$$

Constraint Manifold Equation

- The constraint manifold for a given dyad represents all relative displacements of the dyad links when disconnected from the other two links in a four-bar mechanism.
- An expression for the image space manifold that corresponds to the kinematic constraints emerges when $(X : Y : Z)$, or $(x : y : z)$ from

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & 2X_3X_4 & 2(X_1X_3 - X_2X_4) \\ -2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 + X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

are substituted into

$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0.$$

Constraint Manifold Equation

- The result is the general image space constraint manifold equation:

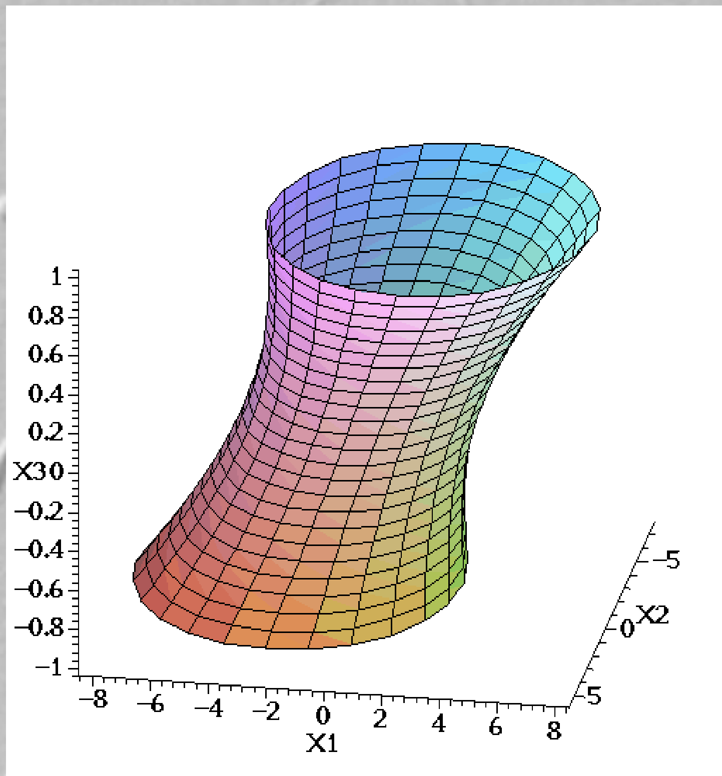
$$CS: K_0(X_1^2+X_2^2)+\frac{1}{4}(K_0[x^2+y^2]+K_3-2[K_1x+K_2y])X_3^2-(K_1+K_0x)X_1X_3+(K_2-K_0y)X_2X_3 \mp$$

$$(K_0y+K_2)X_1 \pm (K_0x+K_1)X_2 \mp (K_1y-K_2x)X_3 +$$

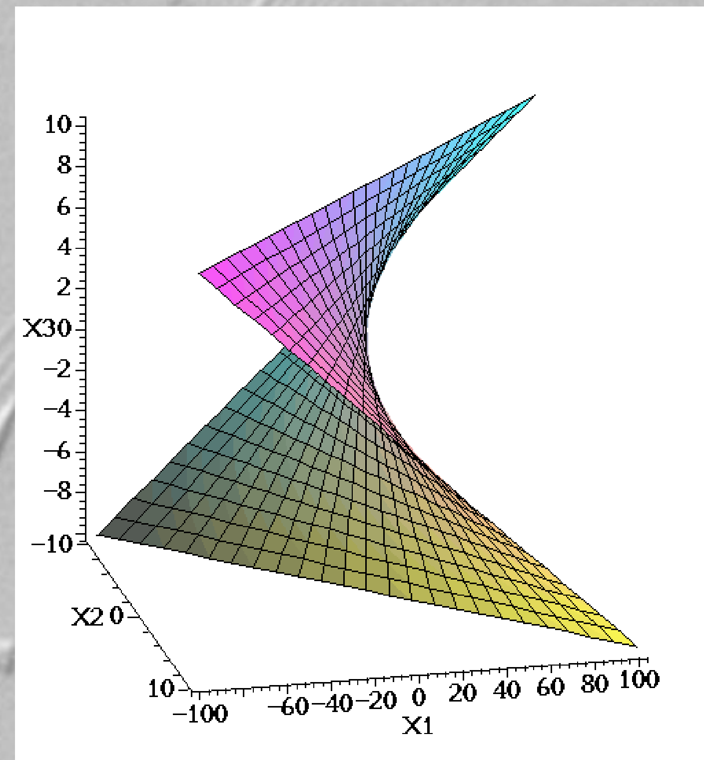
$$\frac{1}{4}(K_0[x^2+y^2]+K_3+2[K_1x+K_2y])=0.$$

- If the kinematic constraint is
  - a fixed point in *E* bound to a circle (K_0=1), or line (K_0=0) in Σ, then ($x : y : z$) are the coordinates of the coupler reference point in *E* and the upper signs apply.
  - a fixed point in Σ bound to a circle (K_0=1), or line (K_0=0) in *E*, then ($X : Y : Z$) are substituted for ($x : y : z$), and the lower signs apply.

Constraint Manifold Equation



K_0 =1: the *CS* is a skew hyperboloid of one sheet (*RR* dyads).



K_0 = 0: *CS* is an hyperbolic paraboloid (*RP* and *PR* dyads).

MECH 5507
Advanced Kinematics

Quadratic Forms and Quadrics

Professor M.J.D. Hayes
Department of Mechanical and
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Quadratic Forms

- The general equation of the second degree in the plane in Cartesian coordinates *x*, *y* is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

- It can be written in matrix form

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2\begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f = 0$$

- The associated quadratic form is

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Quadratic Forms

- The magnitude of the determinant of the 2x2 matrix characterizes the shape of the curve the quadratic equation represents.

$$\text{Let } \det\begin{bmatrix} a & b \\ b & c \end{bmatrix} = ac - b^2 = \Delta$$

- We have the following possibilities:

$$\Delta \begin{Bmatrix} < 0 \\ = 0 \\ > 0 \end{Bmatrix} \Rightarrow \text{the curve is} \begin{Bmatrix} \text{an ellipse, a circle if } a = c \text{ and } b = 0\text{, or a point} \\ \text{a parabola, or 2 parallel lines which may be coincident} \\ \text{a hyperbola, or two intersecting lines} \end{Bmatrix}$$

- Hence, all equations of the second degree in the plane are conic sections, or degenerate conics.

Quadratic Forms

- The general equation of the second degree in the plane can be written symmetrically if homogeneous coordinates are used

$$\text{Let } x_1 = \frac{x}{x_3}, x_2 = \frac{y}{x_3}, x_3 \neq 0.$$

- The general equation becomes

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$$

where $a_{11} = a, a_{12} = b, a_{13} = c, a_{22} = d, a_{23} = e, a_{33} = f$

- Which can be written in a symmetric matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Quadratic Forms

- Of the 9 coefficients, only 6 are distinct.
- Since we can homogenize the 6 coefficients using any one of them, only 5 of the coefficients are independent.
- Thus, in general, 5 given points in the plane define a unique conic, although it may be degenerate.
- Given the coordinates of 5 points, we can solve the resulting system for the 5 independent a_{ij}.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Grassmannian Formulation

- We can derive the implicit equation of a conic given 5 points in another way.
- This way was developed by Hermann Grassmann in the 1840's.
- It involves expanding a determinant, and is called Grassmannian expansion.
- The Grassmannian for a line, given a pair of non-homogeneous point coordinates is L:

$$\mathrm{L} = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

Grassmannian Formulation

- The Grassmannian for a circle, given three sets of homogeneous point coordinates is C:

$$\mathsf{C} = \begin{vmatrix} x^2 + y^2 & xz & yz & z \\ x_1^2 + y_1^2 & x_1 z_1 & y_1 z_1 & z_1 \\ x_2^2 + y_2^2 & x_2 z_2 & y_2 z_2 & z_2 \\ x_3^2 + y_3^2 & x_3 z_3 & y_3 z_3 & z_3 \end{vmatrix}$$

Grassmannian Formulation

- The Grassmannian for a general conic, given five sets of point coordinates is K:

$$
\mathrm{K} = \begin{vmatrix}
x^2 & y^2 & xy & xz & yz & z \\
x_1^2 & y_1^2 & x_1 y_1 & x_1 z_1 & y_1 z_1 & z_1 \\
x_2^2 & y_2^2 & x_2 y_2 & x_2 z_2 & y_2 z_2 & z_2 \\
x_3^2 & y_3^2 & x_3 y_3 & x_3 z_3 & y_3 z_3 & z_3 \\
x_4^2 & y_4^2 & x_4 y_4 & x_4 z_4 & y_4 z_4 & z_4 \\
x_5^2 & y_5^2 & x_5 y_5 & x_5 z_5 & y_5 z_5 & z_5
\end{vmatrix}
$$

Grassmannian Formulation

- The Grassmannian for a general quadric, given nine 3-tupples of point coordinates in *x*, *y*, *z*, or four-tupples of homogeneous coordinates (*x*:*y*:*z*:*w*) is Q:

$$Q = \begin{vmatrix} x^2 & y^2 & z^2 & xy & xz & yz & x & y & z & w \\ x_1^2 & y_1^2 & z_1^2 & x_1 y_1 & x_1 z_1 & y_1 z_1 & x_1 w_1 & y_1 w_1 & z_1 w_1 & w_1 \\ & & & & \vdots & & & & & \\ x_9^2 & y_9^2 & z_9^2 & x_9 y_9 & x_9 z_9 & y_9 z_9 & x_9 w_9 & y_9 w_9 & z_9 w_9 & w_9 \end{vmatrix}$$

Example

- Given three points on a circle (2,1), (1,2), (0,1)

$$C = \begin{vmatrix} x^2 + y^2 & xz & yz & z \\ 5 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix} =$$

$$2x^2 + 2y^2 - 4xz - 4yz + 2z^2 = 0$$

$$x^2 + y^2 - 2xz - 2yz + z^2 = 0$$

Intersection of Two Quadrics

- The curve of intersection of two image space constraint surfaces can be obtained in the following way:
  - We know the curve is 4th order and points on it have one degree of freedom (a one parameter parametric equation).
  - Solve the two implicit surface equations for X_1 and X_2.
  - The solutions, given the nature of the constraint surfaces, will be functions of X_3.
  - Set $X_3 = t$ and substitute into the expressions for X_1 and X_2.
  - If there are multiple solutions, they may be viewed as parametric *factors* in terms of the parameter $X_3 = t$.
- This is illustrated in the following Maple work sheet.

Intersection of Two Quadrics

Maple 6 Worksheet File

SOLVING THE BURMESTER PROBLEM USING KINEMATIC MAPPING

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ASME DETC 2002

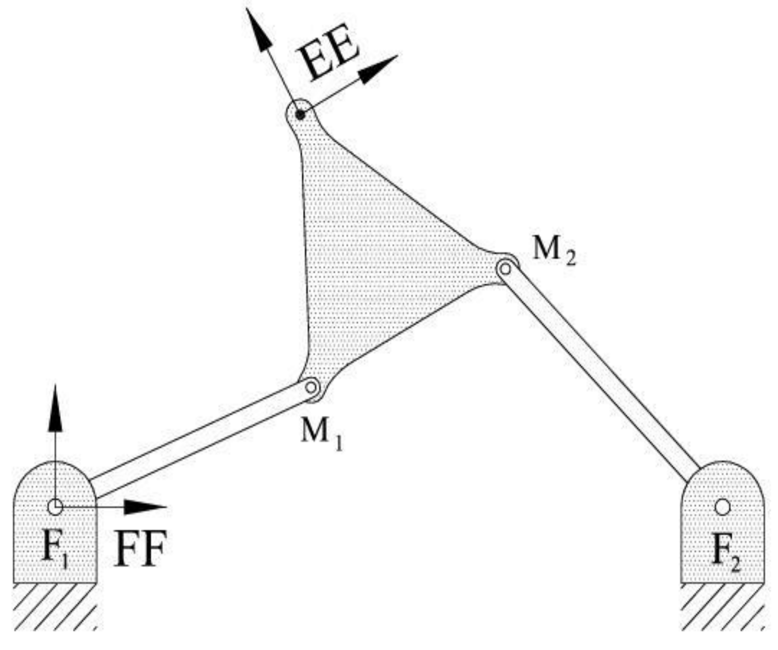
Mechanisms Synthesis and Analysis Symposium - Special Session:

Computer Aided Linkage Design
Montréal, QC

Tuesday October 1, 2002

Five Position Exact Synthesis

- The *five-position Burmester problem* may be stated as:
  - *given five positions of a point on a moving rigid body and the corresponding five orientations of some line on that body, design a four-bar mechanism whose coupler crank pins are located on the moving body and is assemblable upon these five poses.*



- In this example we assume the dyad types we wish to synthesize by setting K_0=1, thereby specifying *RR*-dyads.

Nature of the Constraint Surfaces

- Burmester theory states that five poses are sufficient for exact synthesis of two, or four dyads capable of, when pared, producing a motion that takes a rigid body through exactly the five specified poses.
- This means that five non coplanar points in the image space are enough to determine two, or four dyad constraint surfaces that intersect in a curve containing the five image points.
- This is interesting, because, in general, nine points are required to specify a quadric surface (any function $f(x,y,z)=0$ is a surface):

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

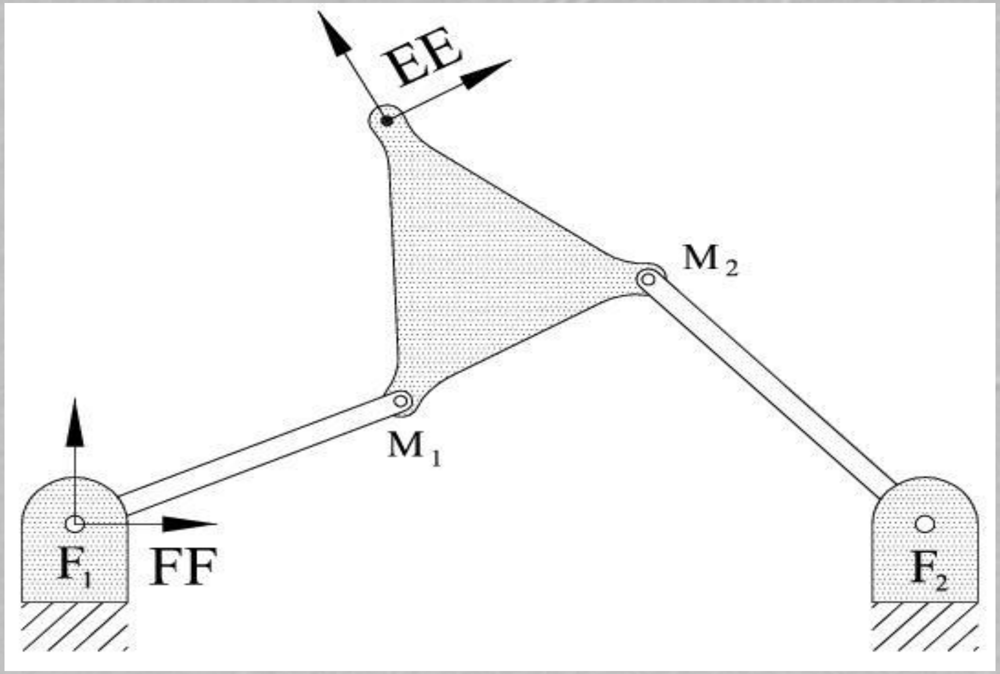
- The equation contains ten *coefficients*; their ratios give nine independent constraints whose values determine the equation.
- It turns out that the special nature of the hyperboloid and hyperbolic paraboloid constraint surfaces represent four constraints on the quadric coefficients; thus five points are sufficient.

Nature of the Constraint Surfaces

- The *RR*-dyad constraint hyperboloids intersect planes parallel to $X_3 = 0$ in circles.
- Thus all constraint hyperboloids contain the image of the imaginary circular points, J_1 and J_2: $(1: \pm i : 0 : 0)$.
- The points J_1 and J_2 are on the line of intersection $X_3 = 0$ and $X_4 = 0$.
- This real line, l, is the axis of a pencil of planes that contain the complex conjugate planes V_1 and V_2, which are defined by $X_3 \pm iX_4 = 0$.
- The *RR*-dyad hyperboloids all have V_1 and V_2 as tangent planes, though not at J_1 and J_2.
- The *PR*- and *RP*-dyad hyperbolic paraboloids contain l as a generator, and therefore also contain J_1 and J_2.
- In addition, V_1 and V_2 are the tangent planes at J_1 and J_2.
- Taken together, these conditions impose four constraints on every constraint surface for *RR*-, *PR*- and *RP*-dyads.
- Thus, only five non coplanar points are required to specify one of these surfaces.

Application to the Burmester Problem

- Goal:
  - determine the moving circle points, M_1 and M_2 of the coupler (revolute centres that move on fixed centred, fixed radii circles as a reference coordinate system, EE, attached to the coupler moves through the given poses).

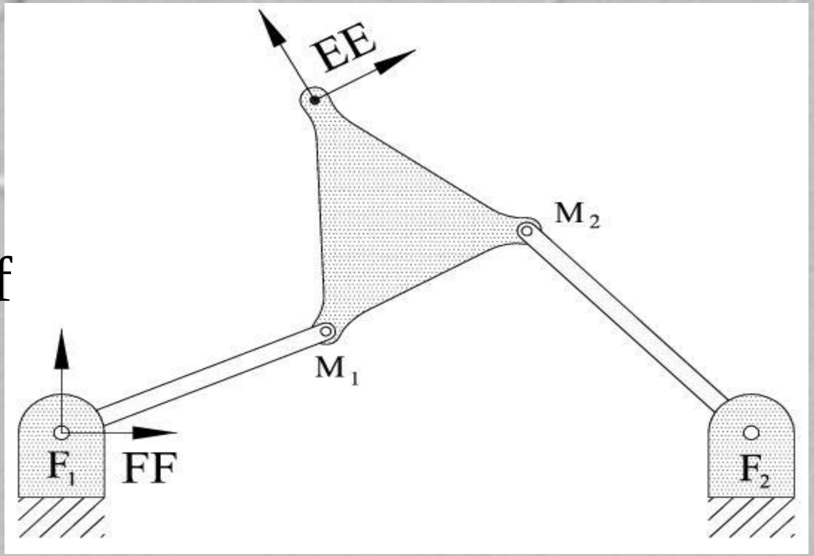


The Five Poses

- To convert specified pose variables a, b, and ϕ to image space coordinates, we first divide through by X_4 to get

$$X_1 = \frac{\left(a\tan(\phi/2) - b\right)}{2}, \quad X_2 = \frac{\left(a + b\tan(\phi/2)\right)}{2}, \quad X_3 = \tan(\phi/2), \quad X_4 = 1.$$

- The five poses are specified as (a_i, b_i, ϕ_i), $i = 1, ..., 5$, the planar coordinates the origin of EE, and orientation all relative to $(0,0,0^o)$ in FF.
- The locations of the origins of FF and EE are arbitrary.



The Five Equations

- We get five simultaneous constraint equations.
- Each represents the constraint surface for a particular dyad.
- This set of equations is expressed in terms of eight variables:
  - *i.* $X_1, X_2, X_3, X_4 = 1$, the dehomogenized coupler pose coordinates in the image space.
  - *ii.* K_1, K_2, K_3, the coefficients of a circle equation ($K_0 = 1$).
  - *iii.* $x, y, z = 1$, coordinates of the moving crank-pin revolute centre, on the coupler, which moves on a circle.
- Since X_1, X_2, X_3, are given, we solve the system for the remaining five variables
  - K_1, K_2, K_3, x, y.

Geometric Interpretation

- The Geometric interpretation is:
  - five given points in space are common to, at most, four RR-dyad hyperboloids of one sheet.
  - If two real solutions result, then all 4*R* mechanism design information is available:

    i. Each circle centre is at $X_C = -K_1,\ Y_C = -K_2$.

    ii. Circle radii are $r^2 = K_3 - (X_C^2 + Y_C^2)$.

    iii. Coupler length is $L^2 = (x_i - x_j)^2 + (y_i - y_j)^2,\ i, j \in \{1,2,3,4\}, i \neq j$.

  - In the case of iii, the subscripts refer to two solutions *i* and *j*.
  - If four real solutions result, the corresponding dyads can be paired in six distinct ways, yielding six 4*R* mechanisms all capable of guiding the coupler through the five specified poses.

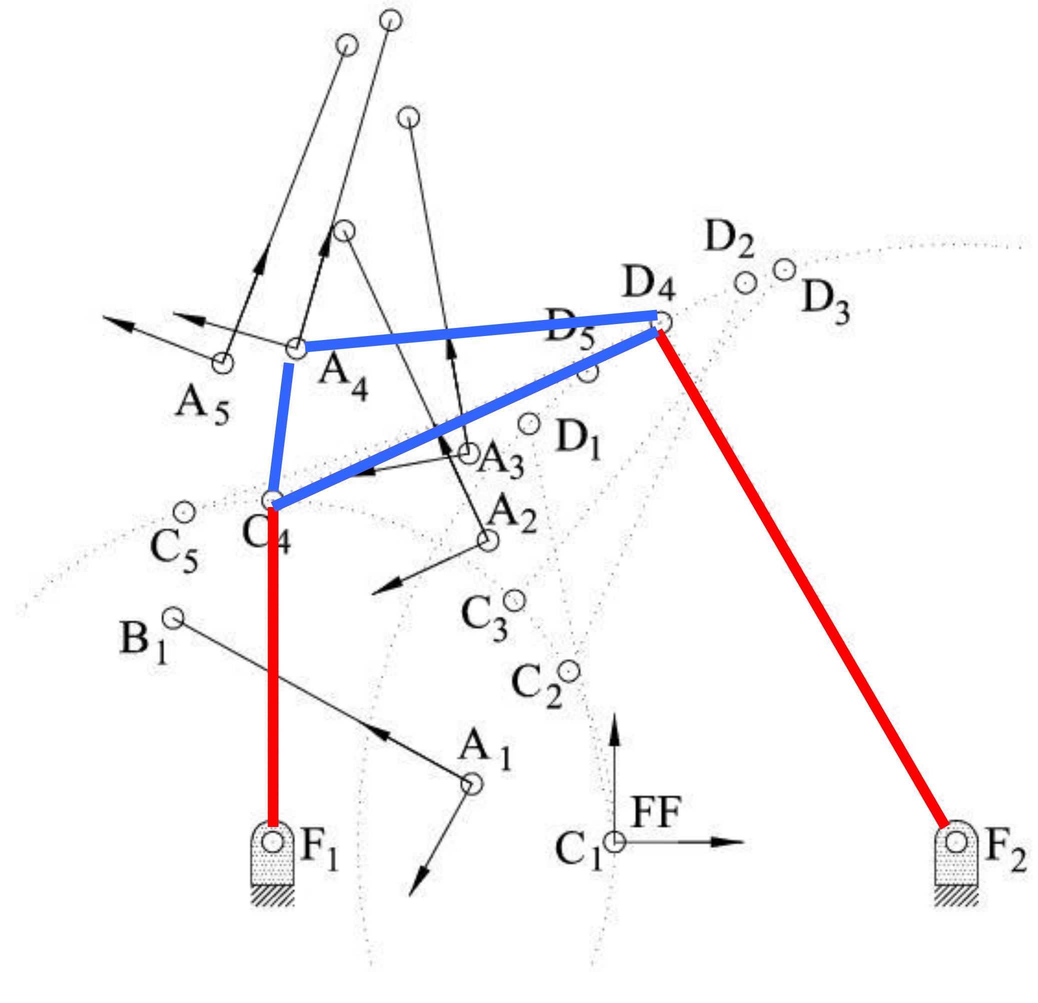
Crank Angles

- To construct the mechanism in its five poses, the crank angles must be determined.
- Take each (x_{i}, y_{i}, $z = 1$), and perform the multiplication for each with the five pose variables in

$$\begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - X_3^2 & -2X_3 & 2(X_1X_3 + X_2) \\ 2X_3 & 1 - X_3^2 & 2(X_2X_3 - X_1) \\ 0 & 0 & X_3^2 + 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix},$$

- The corresponding sets of (X_i, Y_i) are the Cartesian coordinates of the moving *R*-centres expressed in FF, implicitly define the crank angles.
- For a practical design branch continuity must be checked.

Mechanism to Generate Poses



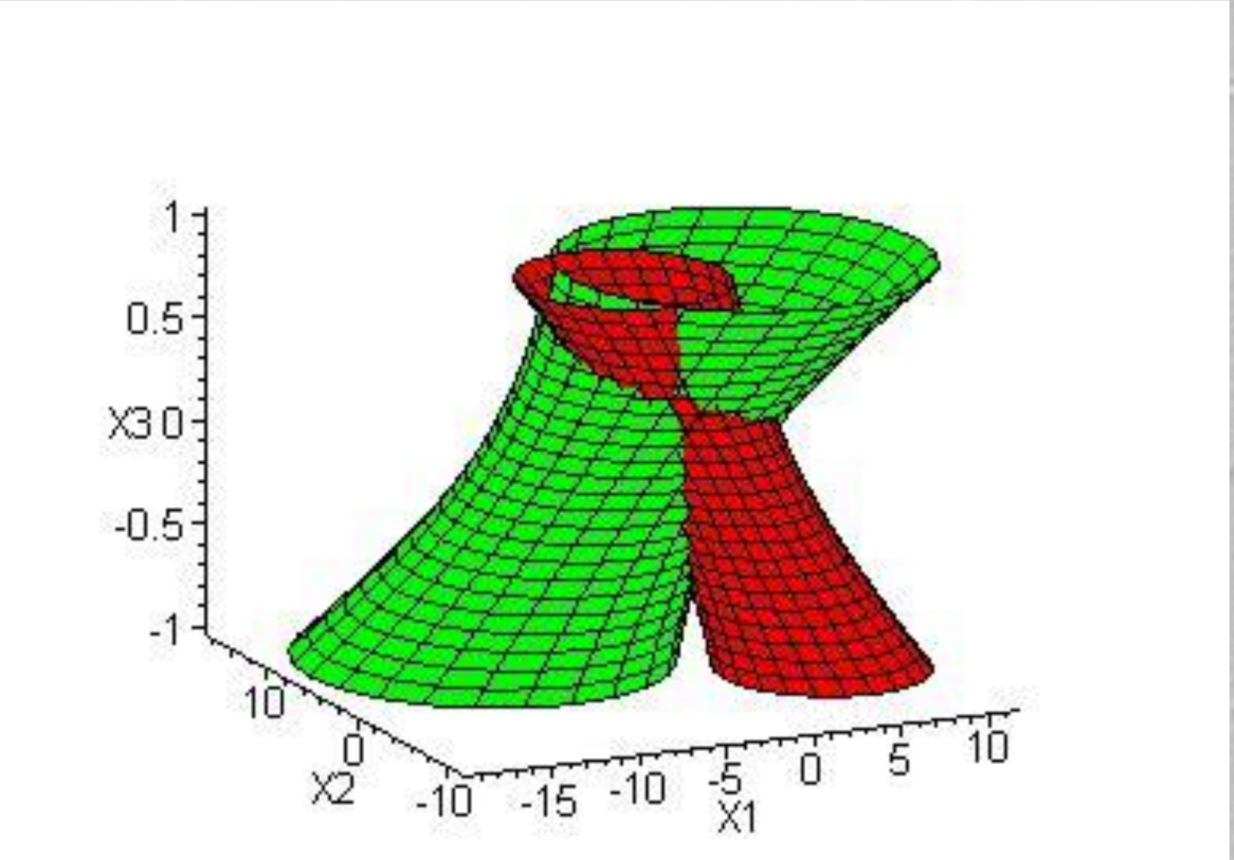
Parameter	Value
F_1	(-8,0)
F_2	(8,0)
F_1F_2	16
F_1C	8
CD	10
DF_2	14

Table 1. THE GENERATING MECHANISM

i^{th} Pose, A_i	a	b	φ (deg)
1	-3.339	1.360	150.94
2	-2.975	7.063	114.94
3	-3.405	9.102	100.22
4	-7.435	11.561	74.07
5	-9.171	11.219	68.65

Table 2. FIVE RIGID BODY POSES IN FF.

The Constraint Hyperboloids



The two constraint hyperboloids for the left and right dyads

Solution

Parameter	Value
F_1	(-7.997,0.001)
F_2	(7.983,-0.023)
F_1F_2	15.980
F_1C	7.999
CD	10.003
DF_2	13.972

Table 4. THE SYNTHESIZED MECHANISM

Parameter	Value
F_1	(-8,0)
F_2	(8,0)
F_1F_2	16
F_1C	8
CD	10
DF_2	14

Table 1. THE GENERATING MECHANISM

MAPLE Implementation

Maple 6 Worksheet
File

Towards Integrated Type and Dimensional Synthesis of Mechanisms for Rigid Body Guidance

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London, ON
June 1 – 4, 2004

Wednesday, June 2, 2004

Integrated Type and Dimensional Synthesis

- Now we try to integrate both *type* and *dimensional* synthesis into one algorithm.
- We shall leave K_0 as an unspecified homogenizing coordinate and solve the five synthesis equations for K_1, K_2, K_3, x, and y in terms of K_0.
- In the solution, the coefficients K_1, K_2, and K_3 will depend on K_0.
- If the constant multiplying K_0 is relatively *very large*, then we will set $K_0 = 0$, and define K_1, K_2, and K_3 as line coordinates proportional to the Grassmann line coordinates:

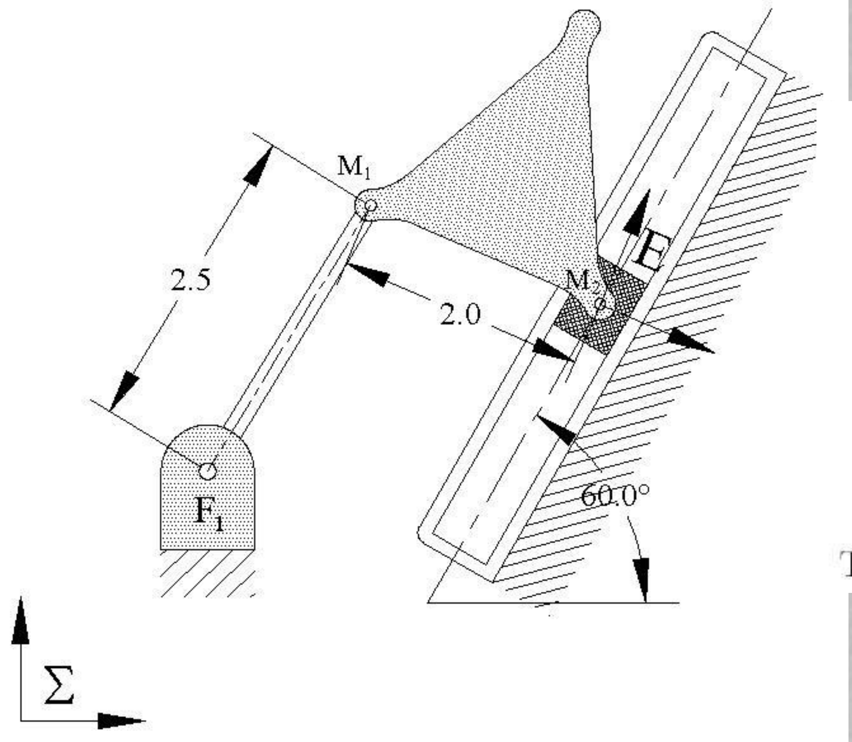
$$[K_1 : K_2 : K_3] = \left[-\frac{1}{2}\sin\xi_\Sigma : \frac{1}{2}\cos\xi_\Sigma : F_{X/\Sigma}\sin\xi_\Sigma - F_{Y/\Sigma}\cos\xi_\Sigma \right].$$

Integrated Type and Dimensional Synthesis

- Otherwise, $K_0 = 1$, and the circle coordinate definitions for K_1, K_2, and K_3 are used:

$$K_0 = \text{arbitrary homogenising constant},$$
$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

Example



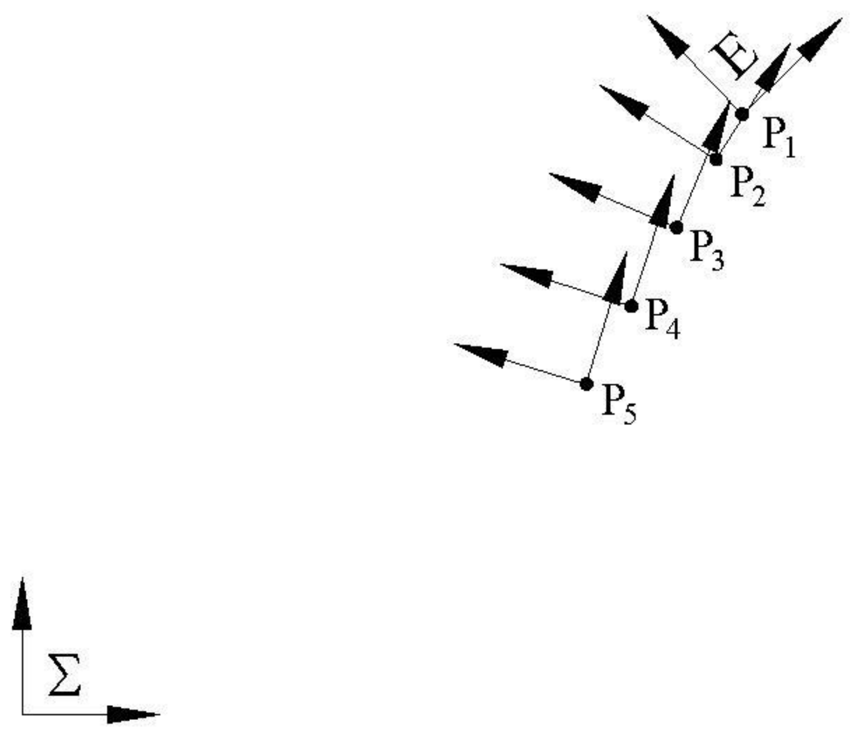
parameter	value
F_1	$(X : Y : Z) = (1.5 : 2 : 1)$
M_1	$(x : y : z) = (-2 : 0 : 1)$
M_2	$(x : y : z) = (0 : 0 : 1)$
M_1M_2	$l = 2$
F_1M_1	$r = 2.5$
P-pair angle	$\vartheta_\Sigma = 60$ (deg)

Table 2: Geometry of the $RRRP$ generating mechanism.

Generated Poses

pose	a	b	φ (deg)
1	5.24080746	4.36781272	43.88348278
2	5.05087057	4.03883237	57.45578356
3	4.76358093	3.54123213	66.99534998
4	4.43453496	2.97130779	72.10014317
5	4.10748142	2.40483444	72.30529428

Table 1: The five desired poses of the $RRRP$ mechanism



- Convert these pose coordinates to image space coordinates $(X_1{:}X_2{:}X_3{:}1)$, and substitute into the general image space constraint manifold equation.
- This yields five polynomial equations in terms of the K_i, x and y.
- Solving for K_1, K_2, K_3, x and y in terms of the homogenizing circle, or line coordinate K_0 yields:

Solutions

Parameter	Surface 1	Surface 2	Surface 3	Surface 4
K_1	$-1.500K_0$	$-4.2909 \times 10^6 K_0$	$-15.6041K_0$	$-8.3011K_0$
K_2	$-2.0000K_0$	$2.4773 \times 10^6 K_0$	$3.4362K_0$	$-5.0837K_0$
K_3	$-2.5801 \times 10^{-6} K_0$	$2.3334 \times 10^7 K_0$	$107.3652K_0$	$93.4290K_0$
x	-2.0000	8.1749×10^{-7}	0.2281	3.7705
y	3.4329×10^{-7}	-1.3214×10^{-6}	-0.7845	-2.0319

Table 3: The constraint surface coefficients.

- At present, heuristics must be used to select an appropriate *value* for K_0 by comparing the relative magnitudes of K_1 and K_2.
- The coefficients for Surfaces 1, 3, and 4 suggest *RR*-dyads when $K_0=1$.
- The rotation centre for Surface 2 is numerically large : (4.3x10^6, -2.5x10^6).
- The crank radius is about 5x10^6.
- This surface should be recomputed as an hyperbolic paraboloid, revealing the corresponding *PR*-dyad.

PR-Dyad

- The reference point with fixed point coordinates in *E* is the rotation centre of the *R*-pair.
- In a *PR*-dyad, it is clear that this point is constrained to be on the line parallel to the direction of translation of the *P*-pair.
- From the Surface 2 coefficients we have $(x,y)=(8.1749\text{x}10^{-7},-1.3214\text{x}10^{-6})$.
- We could transform these coordinates to Σ using one of the specified poses to obtain the required point coordinates, but they are sufficiently close to 0 to assume they are the origin of moving reference frame *E*.
- The angle of the direction of translation of the *P*-pair relative to the X-axis of Σ is ξ_Σ, and is

$$\xi_\Sigma = \arctan\left(\frac{-K_1}{K_2}\right) = \arctan\left(\frac{4.2909\times 10^6 K_0}{2.4773\times 10^6 K_0}\right) = 60.0^o.$$

Dyads

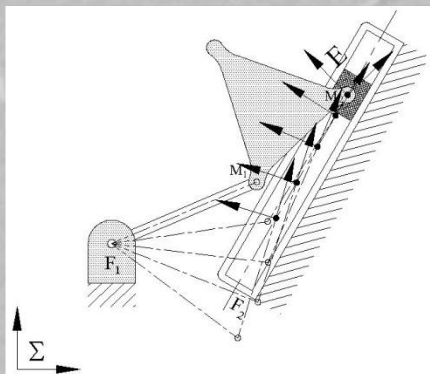
Parameter	Relation	Value
F_1	$(-K_{1_1}, -K_{2_1})$	$(1.500, 2.000)$
M_1	(x_1, y_1)	$(-2.000, 3.4329 \times 10^{-7})$
M_2	(x_2, y_2)	$(8.1749 \times 10^{-7}, -1.3214 \times 10^{-6})$
ϑ_Σ	$\arctan\left(\frac{-K_{1_1}}{K_{2_1}}\right)$	$60.0°$

Table 4: Geometry of one of six synthesized mechanisms that is identical to the generating $RRRP$ linkage in Figure 1.

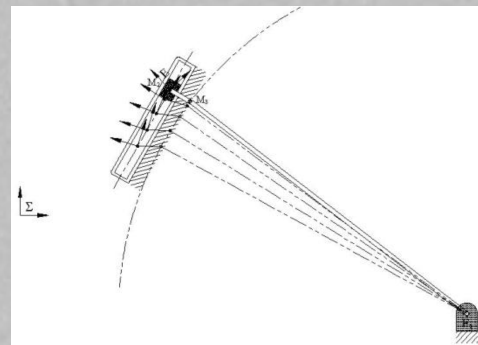
Solution	Dyad surface pairing
1	Dyad 1 - Dyad 2
2	Dyad 2 - Dyad 3
3	Dyad 2 - Dyad 4
4	Dyad 1 - Dyad 3
5	Dyad 1 - Dyad 4
6	Dyad 3 - Dyad 4

Table 5: Dyad pairings yielding the six synthesized mechanisms.

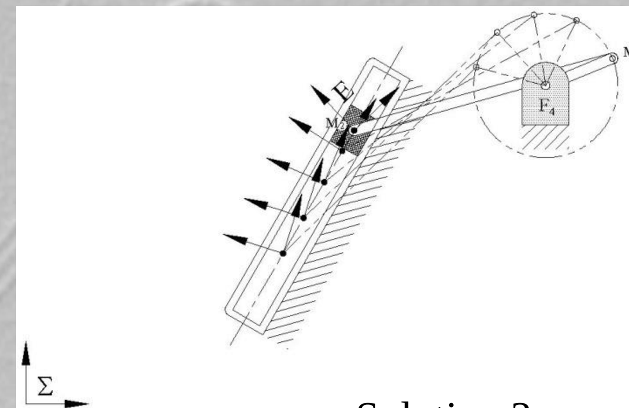
Dyad Pairings



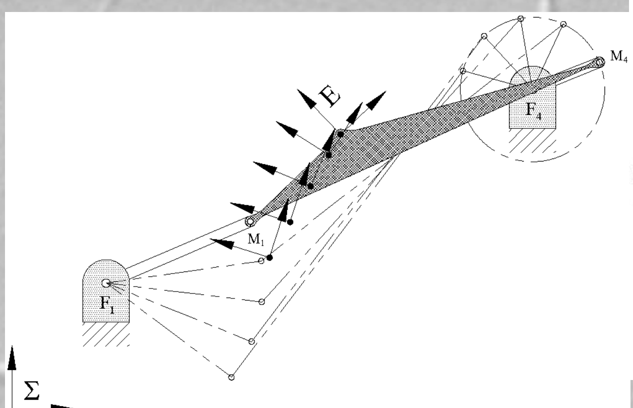
Solution 1



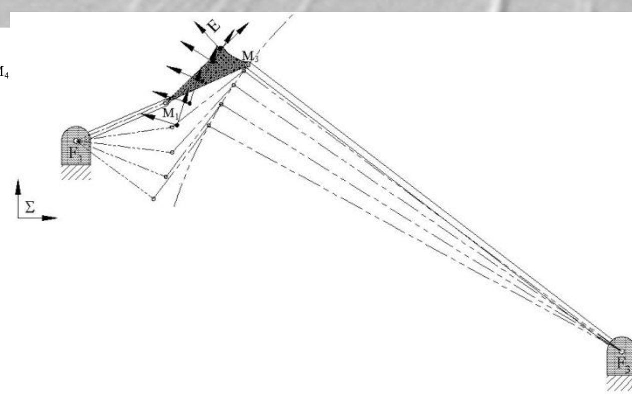
Solution 2



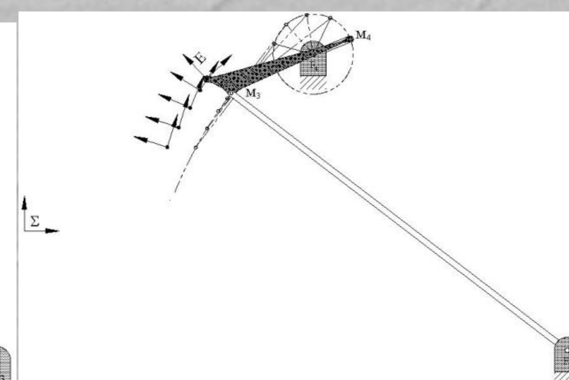
Solution 3



Solution 4



Solution 5



Solution 6

MAPLE Implementation

Maple 6 Worksheet
File

Kinematic Mapping Application to Approximate Type and Dimension Synthesis of Planar Mechanisisms

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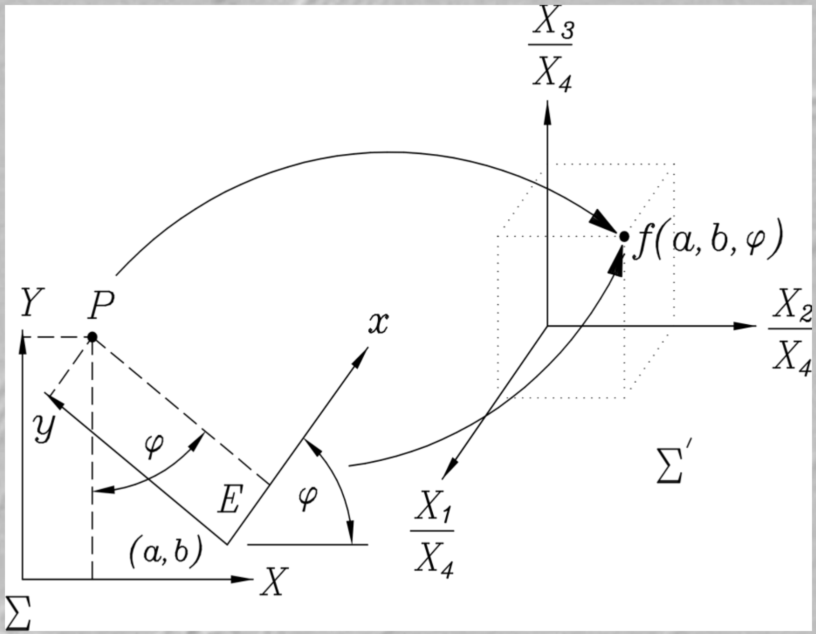
Kinematic Mapping

- The mapping takes distinct poles to distinct points in a 3-D projective image space. It is defined by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} a\sin(\varphi/2) - b\cos(\varphi/2) \\ a\cos(\varphi/2) + b\sin(\varphi/2) \\ 2\sin(\varphi/2) \\ 2\cos(\varphi/2) \end{bmatrix}$$

- Dividing by X_4 normalizes the coordinates:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(a\tan(\varphi/2) - b\right) \\ \frac{1}{2}\left(a + b\tan(\varphi/2)\right) \\ \tan(\varphi/2) \\ 1 \end{bmatrix}$$



- The inverse mapping is:

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2) \end{aligned}$$

Kinematic Mapping

- Using half-angle substitutions and these above relations the basic Euclidean group of planar displacements can be written in terms of the image points

$$\lambda \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- λ being non-zero scaling factors arising from the use of homogeneous coordinates.

Circle and Line Coordinates

- Consider the motion of a fixed point in *E* constrained to move on a fixed circle in Σ, with radius *r*, centred on the homegeneous coordinates $(X_C : Y_C : Z)$ and having the equation

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0,$$

where

$$K_0 = \text{arbitrary homogenising constant.}$$

– If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$

$$K_2 = -Y_C,$$

$$K_3 = K_1^2 + K_2^2 - r^2.$$

– If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

Constraint Manifold Equation

- The constraint manifold for a given dyad represents all relative displacements of the dyad.
- An expression for the image space manifold that corresponds to the kinematic constraints emerges when $(X : Y : Z)$ from

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

are substituted into

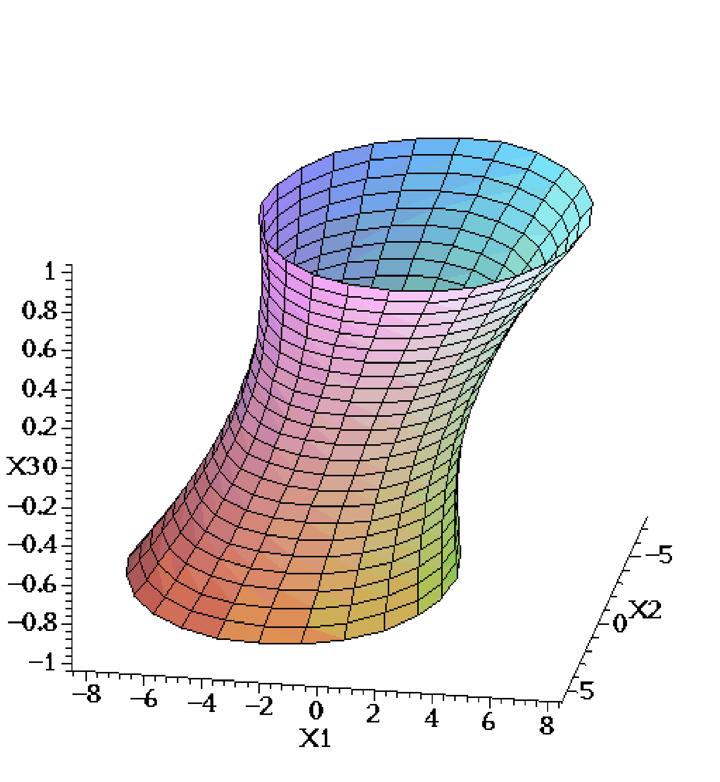
$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0.$$

Constraint Manifold Equation

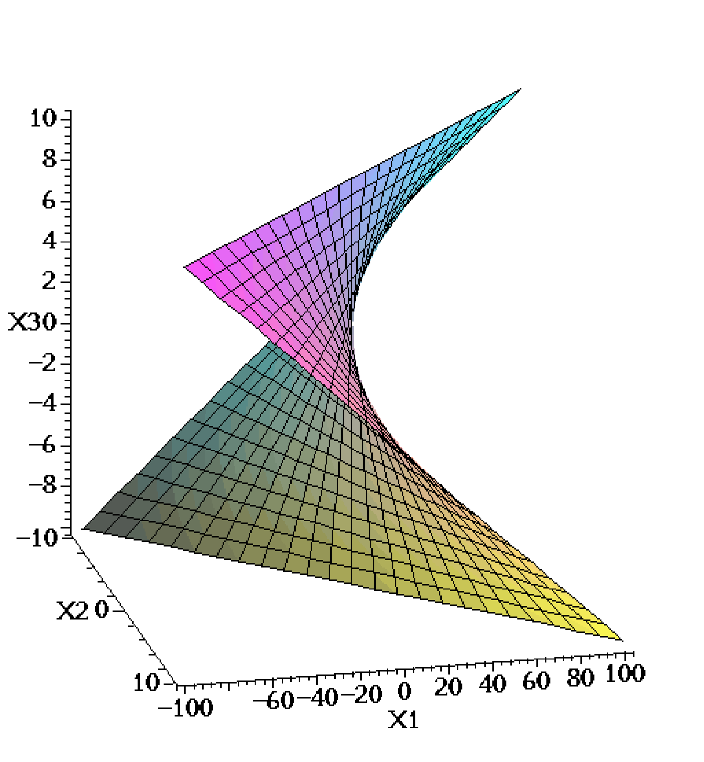
- The result is the general image space constraint manifold equation:

$$CS: K_0(X_1^2+X_2^2)+\frac{1}{4}(K_0[x^2+y^2]+K_3-2[K_1x+K_2y])X_3^2-(K_1+K_0x)X_1X_3+$$
$$(K_2-K_0y)X_2X_3-(K_0y+K_2)X_1+(K_0x+K_1)X_2-(K_1y-K_2x)X_3+$$
$$\frac{1}{4}(K_0[x^2+y^2]+K_3+2[K_1x+K_2y])=0.$$

Constraint Manifold Equation



K_0 =1: the *CS* is a skew hyperboloid of one sheet (*RR* dyads).



K_0 = 0: *CS* is an hyperbolic paraboloid (*RP* and *PR* dyads).

SVD

- Any *m* x *n* matrix **C** can be decomposed into the product

$$\mathbf{C}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^T_{n\times n}$$

  - where **U** is an orthogonal matrix ($\mathbf{U}\mathbf{U}^T=\mathbf{I}$),
  - the uppermost $n \times n$ elements of **S** are a diagonal matrix whose elements are the singular values of **C**,
  - **V** is an orthogonal matrix ($\mathbf{V}\mathbf{V}^T=\mathbf{I}$).
- The singular values, s_i, of **C** are related to its eigenvalues, λ_i. If **C** is rectangular $\mathbf{C}^T\mathbf{C}$ is positive semidefinite with non-negative eigenvalues:

$$(\mathbf{C}^T\mathbf{C})\mathbf{x} = \lambda\mathbf{k} \;\; \Rightarrow \;\; \left(\mathbf{C}^T\mathbf{C} - \lambda\mathbf{I}\right)\mathbf{k} = 0$$

and $s_i = \sqrt{\lambda_i}$

SVD

- SVD explicitly constructs orthonormal bases for the nullspace and range of a matrix.
  - The columns of **U** whose same-numbered elements s_i are non-zero are an orthonormal set of basis vectors spanning the range of **C**.
  - The columns of **V** whose same-numbered elements s_i are zero are an orthonormal set of basis vectors spanning the nullspace of **C**.
- If $\mathbf{C}_{m\times n}$ does not have full column rank then the last n-rank(**C**) columns of **V** span the nullspace of **C**.
- Any of these columns, in any linear combination, is a non-trivial solution to

$$\mathbf{Ck}=\mathbf{0}$$

Aside: Line and circle feature extraction

- Not specifying a value for K_0 gives a homogeneous linear equation in the K_i:

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0.$$

- It is homogeneous in the projective geometric sense, and homogeneous in the linear algebraic sense in that the constant term is 0:

$$\mathbf{kX}^T=0.$$

- Four points in the plane yields the following homogeneous system of linear equations:

$$\mathbf{Xk}=\begin{bmatrix} X_1^2+Y_1^2 & 2X_1Z & 2Y_1Z & Z^2 \\ X_2^2+Y_2^2 & 2X_2Z & 2Y_2Z & Z^2 \\ X_3^2+Y_3^2 & 2X_3Z & 2Y_3Z & Z^2 \\ X_4^2+Y_4^2 & 2X_4Z & 2Y_4Z & Z^2 \end{bmatrix}\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}=\mathbf{0}$$

Line and circle feature extraction

- In the general case where **X** has full rank the system has either
  - only the trivial solution, **k**=**0**, or
  - infinitely many nontrivial solutions in addition to the trivial solution.
- Not very useful for feature identification if **k** characterizes the feature.
- However, if the points are all on a line or a circle, then **X** becomes rank deficient by 1.
- In other words, **X** acquires a nullity of 1: the dimension of the nullspace is 1 and is spanned by a single basis vector.
- Since the singular values are lower bounded by 0 and arranged in descending order on the diagonal of **S** by the SVD algorithm a nontrivial solution for **k** is the same numbered column in **V** corresponding to s_i=0.
- This is true for any $\mathbf{X}_{m\text{x}4}$, where m≥4, having a nullity of 1.

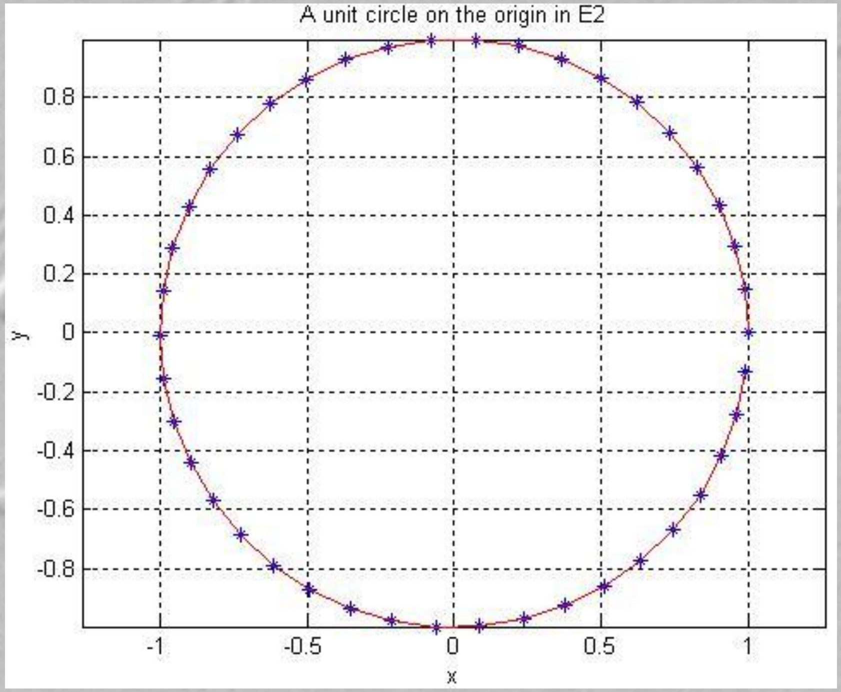
Points Falling Exactly on a Circle

- Given 42 points falling exactly on the unit circle centred on the origin generated by the parametric equations

$$X = r\cos\vartheta$$
$$Y = r\sin\vartheta$$

- This gives rank($\mathbf{X}_{42x4}$)=3
- We have s_4=0 and look at the 4[th] column of **V**:

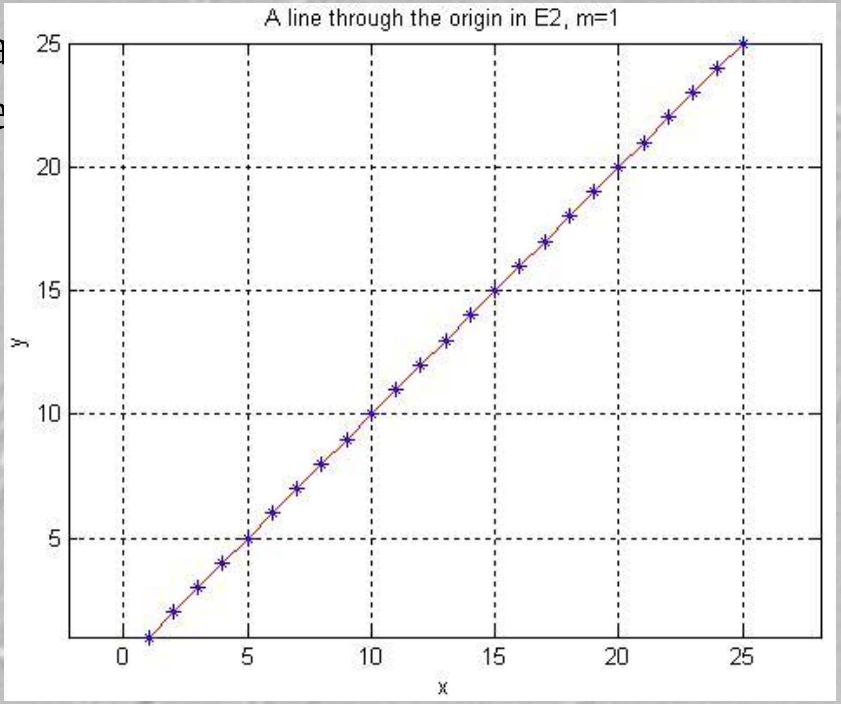
$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \text{circle} \\ -X_c \\ -Y_c \\ K_1^2 + K_2^2 - r^2 \end{bmatrix} \Rightarrow \begin{bmatrix} X_c \\ Y_c \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Points Falling Exactly on a Line

- Given 25 points falling exactly on a line through the origin having slope *m*=1, generated by the parametric equations

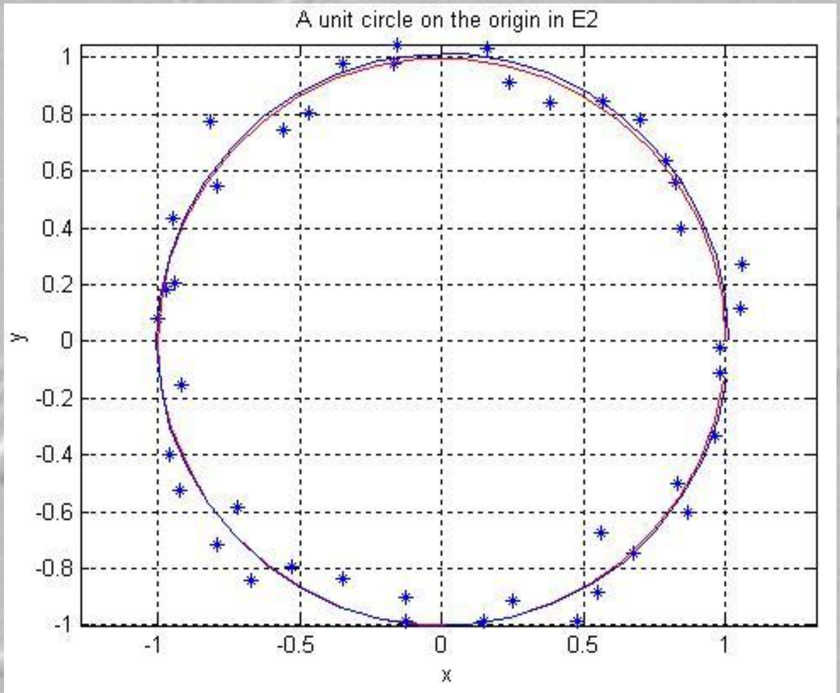
$$X = t$$
$$Y = t$$



- This gives rank($\mathbf{X}_{25x4}$)=3
- We have s_4=0 and look at the 4[th] column of **V**:

$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7071 \\ -0.7071 \\ 0 \end{bmatrix} = \begin{bmatrix} \text{line} \\ -\frac{1}{2}\sin\xi \\ \frac{1}{2}\cos\xi \\ X\sin\xi - Y\cos\xi \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

Points Falling Approximately on a Circle



- Given 42 points falling approximately on the unit circle centred on the origin generated by the parametric equations

$$X = r\cos\vartheta + \text{noise}$$

$$Y = r\sin\vartheta + \text{noise}$$

- This gives rank($\mathbf{X}_{42x4}$)=4 and cond($\mathbf{X}_{42x4}$)=225.2.
- Still,when we look at **V**(:,4):

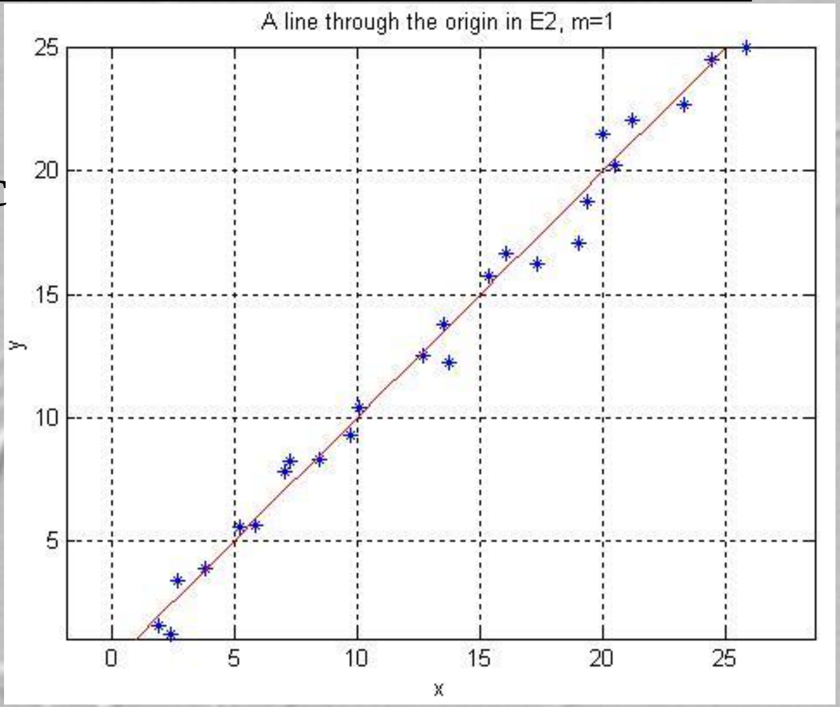
$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.00047 \\ 0.00027 \\ -0.99746 \end{bmatrix} = \begin{bmatrix} \text{circle} \\ -X_c \\ -Y_c \\ K_1^2 + K_2^2 - r^2 \end{bmatrix} \Rightarrow \begin{bmatrix} X_c \\ Y_c \\ r \end{bmatrix} = \begin{bmatrix} 0.00047 \\ -0.00027 \\ 0.99746 \end{bmatrix}$$

Points Falling Approximately on a Line

- Given 25 points falling approximately on a line through the origin having slope *m*=1, generated by the parametric equations

$$X = t + \text{noise}$$

$$Y = t + \text{noise}$$



- This gives rank($\mathbf{X}_{25x4}$)=4 and cond($\mathbf{X}_{25x4}$)=5448.6
- Still,when we look at $\mathbf{V}(:,4)$:

$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} -0.00068 \\ 0.71156 \\ -0.69558 \\ -0.00099 \end{bmatrix} = \begin{bmatrix} \text{approximate line} \\ -\frac{1}{2}\sin\xi \\ \frac{1}{2}\cos\xi \\ X\sin\xi - Y\cos\xi \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ X \\ Y \end{bmatrix} \cong \begin{bmatrix} 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

Approximate Mechanism Synthesis

- To exploit the ability of SVD to construct the basis vectors spanning the nullspace of the homogeneous system of synthesis equations **Ck=0**, we must rearrange the terms in the general constraint surface equation, and for now, restrict ourselves to RR- and PR-dyads.
- We obtain a constraint equation linear in the surface shape parameters K_0, K_1, K_2, K_3, and products with x and y:

$$\left[\frac{1}{4}(X_3+1)x^2 + (X_2 - X_1X_3)x + \frac{1}{4}(X_3+1)y^2 - (X_1 + X_2X_3)y + X_2^2 + X_1^2\right]K_0 +$$

$$\left[\frac{1}{2}(1-X_3^2)x - X_3y + X_1X_3 + X_2\right]K_1 + \left[X_3x + \frac{1}{2}(1-X_3^2)y - X_1 + X_2X_3\right]K_2 + \frac{1}{4}(X_3^2+1)K_3 = 0.$$

Approximate Mechanism Synthesis

- There are 12 terms. The X_i are assembled into the m x 12 coefficient matrix **C**.
- The corresponding vector **k** of shape parameters is:

$$\begin{bmatrix} K_0 & K_1 & K_2 & K_3 & K_0 x & K_0 y & K_0 x^2 & K_0 y^2 & K_1 x & K_1 y & K_2 x & K_2 y \end{bmatrix}^{\mathrm{T}}$$

- Several of the elements of **k** have identical coefficients in **C**:
  - $\frac{1}{4}\left(1+X_3^2\right)$ is the coefficient of K_0x^2, K_0y^2, and K_3.
  - $\frac{1}{2}\left(1-X_3^2\right)$ is the coefficient of K_1x and K_2y.
  - X_3 is the coefficient of K_2x and K_1y.

Approximate Mechanism Synthesis

- The like terms may be combined yielding an *m* x 8 coefficient matrix **C** whose elements are:

$$\left[X_1^2+X_2^2 \quad X_2+X_1X_3 \quad X_2X_3-X_1 \quad X_2-X_1X_3 \quad -\left(X_1+X_2X_3\right) \quad \frac{1}{4}\left(1+X_3^2\right) \quad \frac{1}{2}\left(1-X_3^2\right) \quad X_3 \right]$$

- The corresponding 8 x 1 vector **k** of shape parameters is:

$$\left[K_0 \quad K_1 \quad K_2 \quad K_0x \quad K_0y \quad K_0\left(x^2+y^2\right)+K_3 \quad \left(K_1x+K_2y\right) \quad \left(K_2x-K_1y\right) \right]^{\mathrm{T}}$$

- We now have a system of *m* homogeneous equations in the form

$$\mathbf{Ck}=\mathbf{0}$$

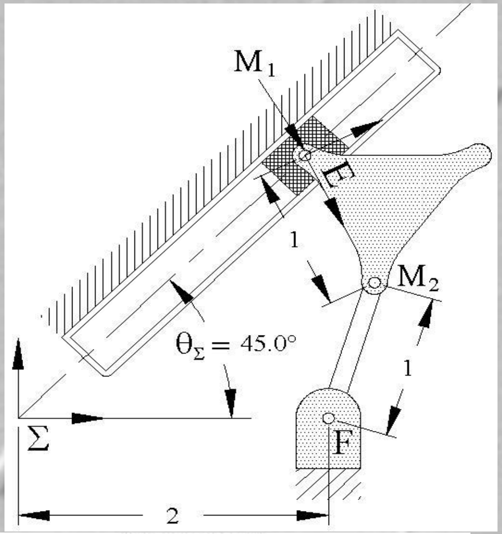
Approximate Mechanism Synthesis

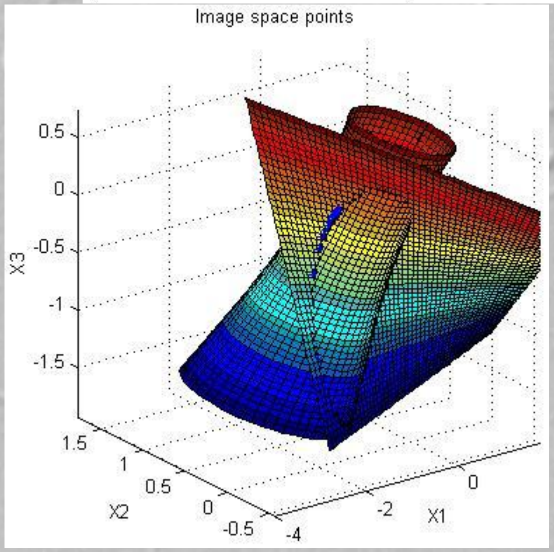
- We obtain the following correspondence between rank(**C**), the mechanical constraints, and the order of the coupler curve:

rank(**C**)	constraint	coupler curve order
8	general planar motion	??
6	two RR-dyads	6
6	one PR-, one RR-dyad	4
5	two PR-dyads	2

- In general, rank(**C**) = 8, with O_E on neither a line or circle.
- Practical application of the approach will require fitting constraint surfaces to their approximate curve of intersection, which means rank(**C**) = 8.
- We will have to approximate **C** by matrices of lower rank.
- To start we will investigate Eckart-Young-Mirsky theory.

Example





- An exploratory experiment was devised.
- A PRRR mechanism was used to generate a set of 20 coupler positions and orientations using the origin of E, given by the coordinates (*a*,*b*), as the coupler point, and taking its orientation to be that of the coupler.
- The positions range from (2,1) to (3,2), and the orientations from -5° to -90º.
- The range of motion of the PR- and RR-dyads map to a hyperbolic paraboloid and hyperboloid of one sheet, respectively.
- These quadrics intersect in a spatial quartic, such that rank(**C**) = 6.

Example

- When the rank of a 20 x 8 matrix is deficient by 2, then 2 columns are linear combinations of the remaining 6.
- The column vectors **V**(:,6) and **V**(:,7) in the SVD of **C** span its nullspace.
- Any linear combination **V**(:,6) + λ**V**(:,7) is a solution.
- But, we can regard this in a different way.
- We can combine these columns of **C** and corresponding elements of **k**.
- The rank of **C** is invariant under this process.
- We obtain two different 20 x 7 coefficient matrices possessing rank = 6.
- The resulting two nullspace vectors represent the generating PR-, and RR-dyads, exactly.

PR-Dyad Synthesis

- To extract the PR-dyad we set K_0=0.
- Recall

$$\mathbf{k} = \begin{bmatrix} K_0 & K_1 & K_2 & K_0 x & K_0 y & K_0(x^2+y^2)+K_3 & (K_1 x + K_2 y) & (K_2 x - K_1 y) \end{bmatrix}^T$$

- In the system **Ck** = **0** we can add columns 4 and 5 of **C** because K_0=0.
- The resulting 20 x 7 matrix **C** possesses rank 6.
- The 7th column of the **V** matrix that results from the SVD of **C** yields **k** that exactly represents the constraint surface for the generating PR-dyad.

RR-Dyad Synthesis

- To extract the RR-dyad we add columns 2 and 3 of **C**.
- This can be done when $(X_1 - X_2X_3)/(X_1X_3 + X_2)$ has the same scalar value for every, X_1, X_2, and X_3 in the pose data.
- The scalar is the ratio K_1/K_2 of the PR-dyad parameters.
- This happens only when PR-dyad design parameters contain

$$K_3 = x = y = 0$$

- In this case the hyperbolic paraboloid has the equation

$$K_1\left(X_1X_3 + X_2\right) + K_2\left(X_2X_3 - X_1\right) = 0$$

- The curve of intersection with any RR-dyad constraint hyperboloid will be symmetric functions of X_3 in X_1 and X_2.
- An image space curve with rank(**C**) = 6 but PR-dyad design parameters

$$K_3 \neq x \neq y \neq 0$$

can always be transformed to one symmetric in X_1 and X_2.

Results

Table 1. Nullspace vectors obtained by adding two columns of $\mathcal{C}$, and same-numbered elements of $\boldsymbol{\kappa}$

Column 4+5	*Value*	*Column 2+3*	*Value*	*Value/K_0*
K_0	0	K_0	-0.2085	1
K_1	0.7071	$K_1 + K_2$	0.2085	-1
K_2	-0.7071	$K_0 x$	-0.2085	1
$K_0(x+y)$	0	$K_0 y$	0	0
$K_0(x^2+y^2)+K_3$	0	$K_0(x^2+y^2)+K_3$	-0.8340	4
$K1x + K2y$	0	$K1x + K2y$	0.4170	-2
$K2x - K1y$	0	$K2x - K1y$	0	0

Table 2. Generating mechanism shape parameters.

Parameter	*PR-dyad*	*RR-dyad*
K_0	0	1
K_1	-1	-2
K_2	1	0
K_3	0	3
x	0	1
y	0	0

Conclusions and Future Work

- We have presented preliminary results that will be used in the development of an algorithm combining type and dimensional synthesis of planar mechanisms for *n*-pose rigid body guidance.
- This approach stands to offer the designer *all* possible linkages that can attain the desired poses, not just 4R's and not just slider-cranks, but *all* four-bar linkages.
- The results are preliminary, and not without unresolved conceptual issues.
  - Cope with *noise*: random noise greater than 0.01% is problematic.
  - Establish how to proceed with 4R mechanisms.
  - For the general approximate case with rank($\mathbf{C}$) = 8, determine how to approximate $\mathbf{C}$ with lower rank matrices.
  - Establish optimization criteria.
  - Investigate meaningful metrics in the kinematic mapping image space.

Integrated Type And Dimensional Synthesis of Planar Four-Bar Mechanisms

© Tim J. Luu[1] and M.J.D. Hayes[2]

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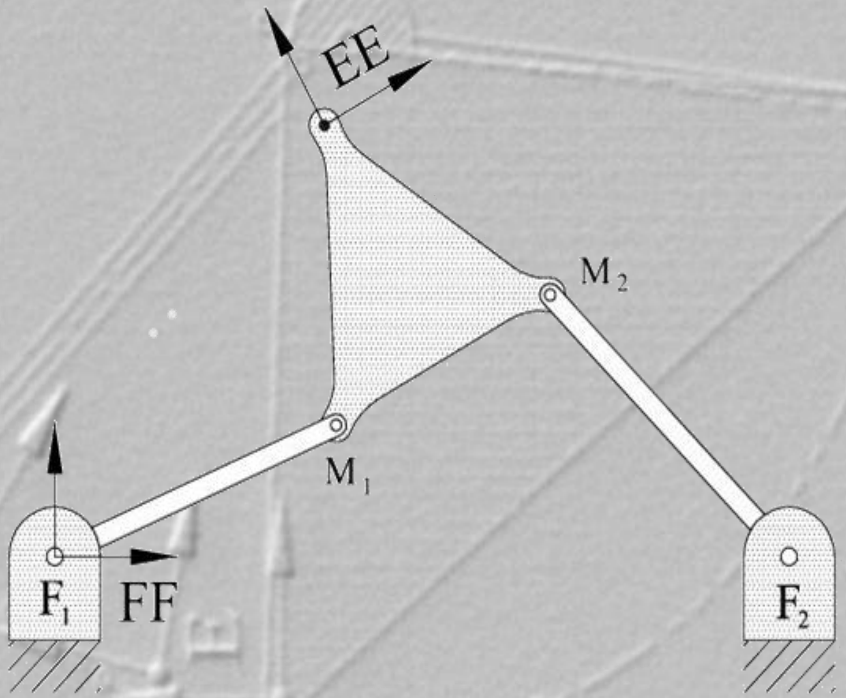
Five Position Exact Synthesis

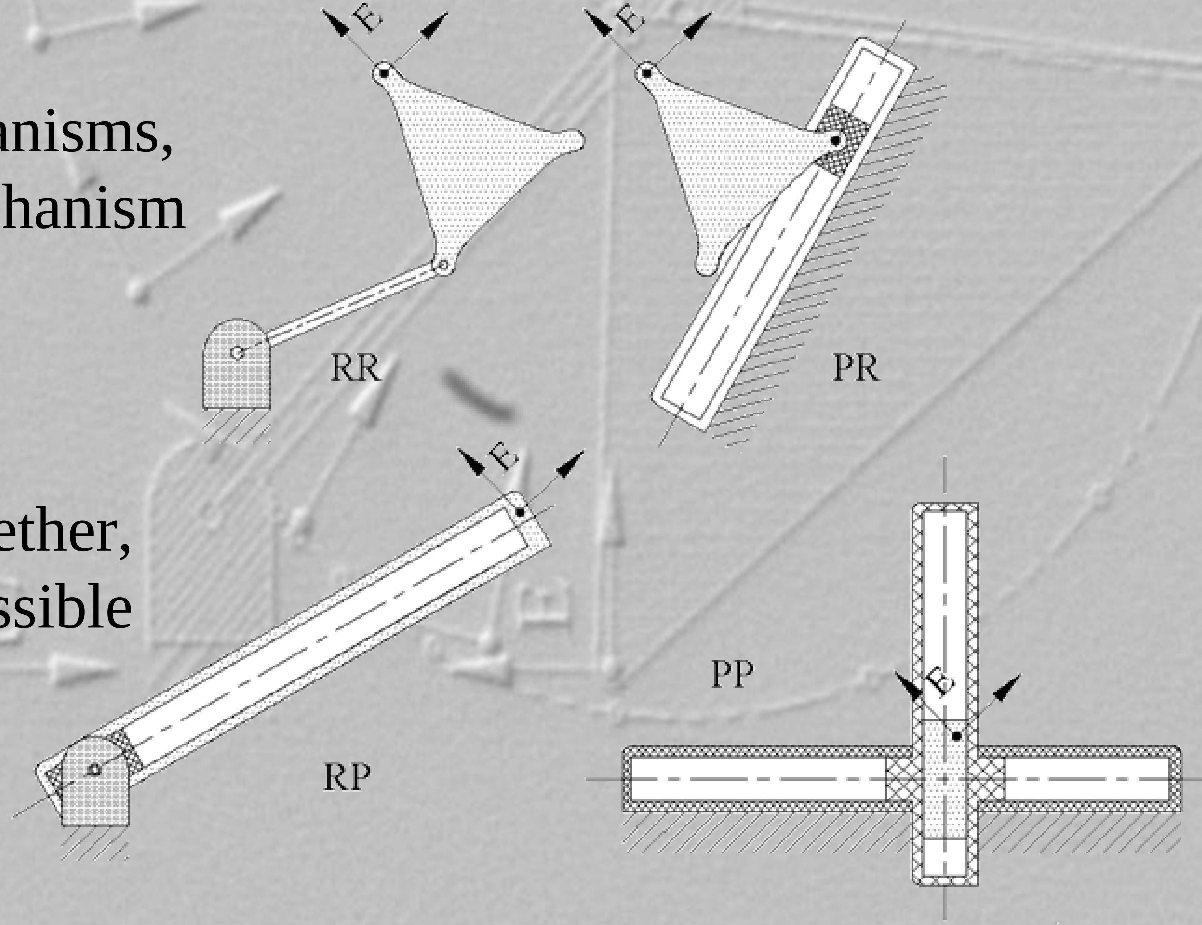
- The *five-position Burmester problem* may be stated as:
  - *given five positions of a point on a moving rigid body and the corresponding five orientations of some line on that body, design a four-bar mechanism that can move the rigid body exactly through these five poses.*



- In general, exact dimensional synthesis for rigid body guidance assumes a mechanism type (4R, slider-crank, elliptical trammel, et c.).
- Our aim is to develop an algorithm that integrates both type and approximate dimensional synthesis for $n > 5$ poses.

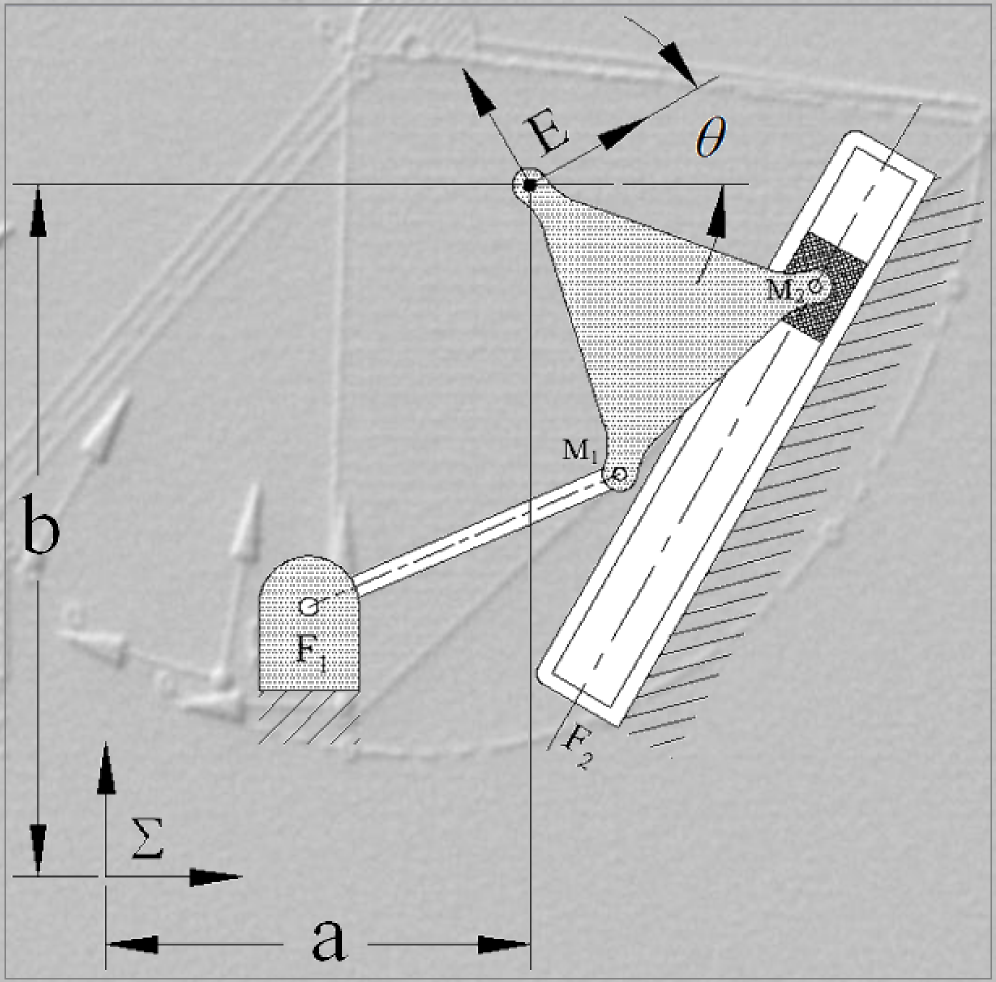
Type Synthesis

- For planar mechanisms, two types of mechanism constraints:
  - Prismatic (P);
  - Revolute (R).
- When paired together, there are four possible dyad types.



Dyad Constraints

- Dyads are connected through the coupler link at points M_1 and M_2.
  - RR – a fixed point in *E* forced to move on a fixed circle in Σ.
  - PR – a fixed point in *E* forced to move on a fixed line in Σ.
  - RP – a fixed line in *E* forced to move on a fixed point in Σ.
  - PP – a fixed line in *E* forced to move in the direction of a fixed line in Σ.

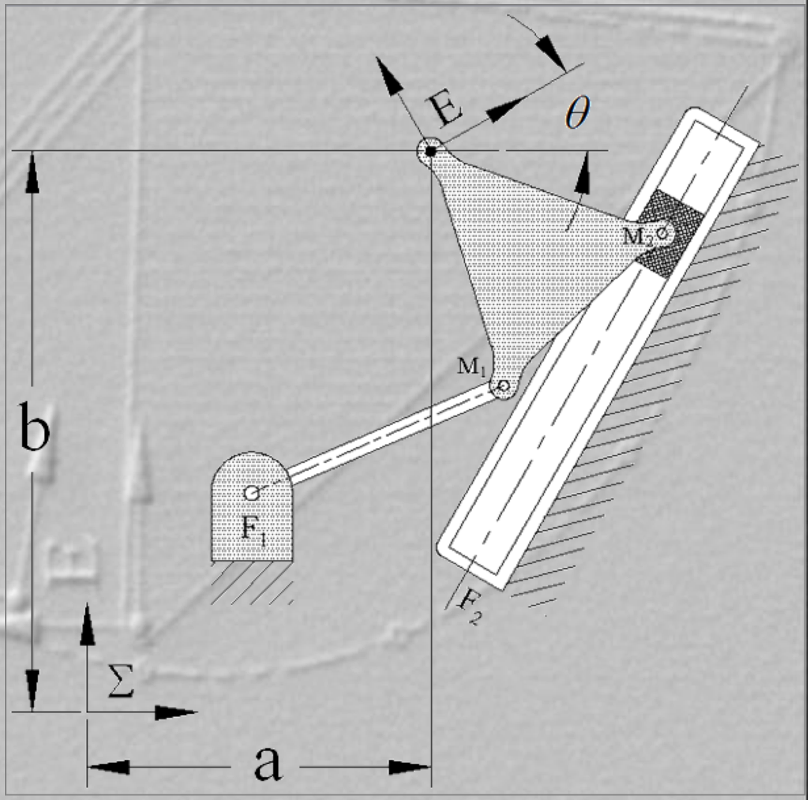


Kinematic Constraints

- Three parameters, a, b and θ describe a planar displacement of E with respect to Σ.
- The coordinates of a point in E can be mapped to those of Σ in terms of a, b and θ:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  - (x:y:z): homogeneous coordinates of a point in E.
  - (X:Y:Z): homogeneous coordinates of the same point in Σ.
  - (a,b): Cartesian coordinates of O_E in Σ.
  - θ: rotation angle from X- to x-axis, positive sense CCW.



Circle and Line Coordinates

- Consider the motion of a fixed point in *E* constrained to move on a fixed circle in Σ, with radius *r*, centred on the homegeneous coordinates $(X_C : Y_C : Z)$ and having the equation

$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0,$$

where

$$K_0 = \text{arbitrary homogenising constant.}$$

  – If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

  – If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

Circle / Line Coordinates

Circle / Line Equation:

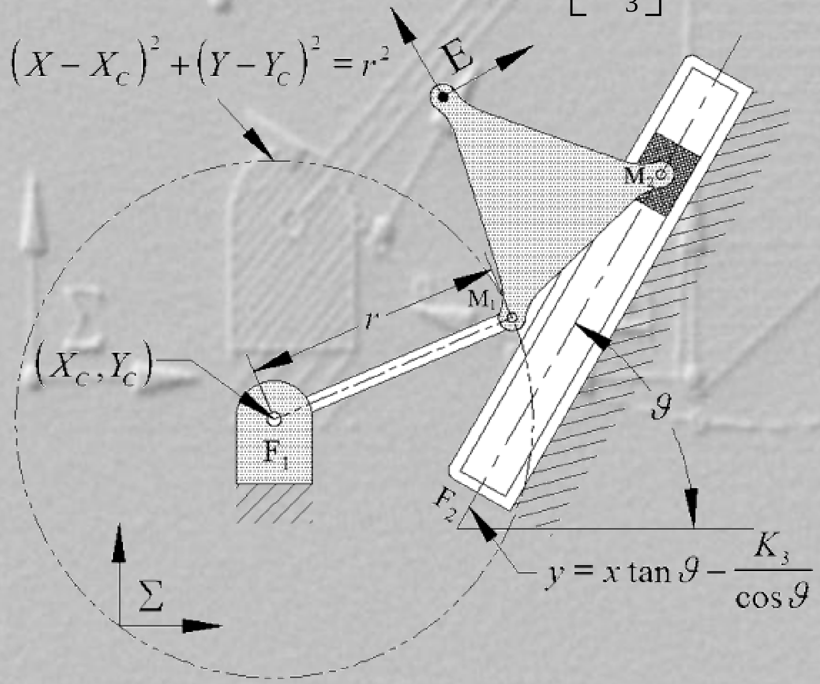
Ignoring infinitely distant coupler attachment points set $Z = z = 1$

$$\mathbf{Ck} = \begin{bmatrix} X^2 + Y^2 & 2X & 2Y & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$$

K_0 acts as a binary switch between circle and line coordinates

Circle Coordinates

$$K_0 = 1$$
$$K_1 = -X_C$$
$$K_2 = -Y_C$$
$$K_3 = K_1^2 + K_2^2 - r^2$$



Line Coordinates

$$K_0 = 0$$
$$K_1 = -\frac{\sin\vartheta_\Sigma}{2}$$
$$K_2 = \frac{\cos\vartheta_\Sigma}{2}$$
$$K_3 = x\sin\vartheta_\Sigma - y\cos\vartheta_\Sigma$$

Reference Frame Correlation

Applying $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ to $\mathbf{Ck} = \begin{bmatrix} X^2 + Y^2 & 2X & 2Y & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$

yields

$$\mathbf{Ck} = \begin{bmatrix} (x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \\ 2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \\ 2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \\ 1 \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}$$

- Prescribing $n > 5$ poses makes **C** an n x 4 matrix.
- **a**, **b**, and **θ** are the specified poses of *E* described in Σ.

Approximate Synthesis for n Points

For *n* poses:

$$\mathbf{Ck} = \begin{bmatrix} (x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \\ 2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \\ 2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \\ 1 \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}$$

- The only two unknowns in **C** are the coordinates *x* and *y* of the coupler attachment points expressed in *E*.
- For non-trivial **k** to exist satisfying **Ck=0**, then **C** must be rank deficient.
- The task is to find values for *x* and *y* that render **C** the most ill-conditioned.

Matrix Conditioning

The condition number of a matrix is defined to be:

$$\kappa \equiv \frac{\sigma_{MAX}}{\sigma_{MIN}},\ 1 \le \kappa \le \infty$$

A more convenient representation is:

$$\gamma \equiv \frac{1}{\kappa},\ 0 \le \kappa \le 1$$

γ is bounded both from above and below.

Choose *x* and *y* in matrix **C** such that γ is minimized.

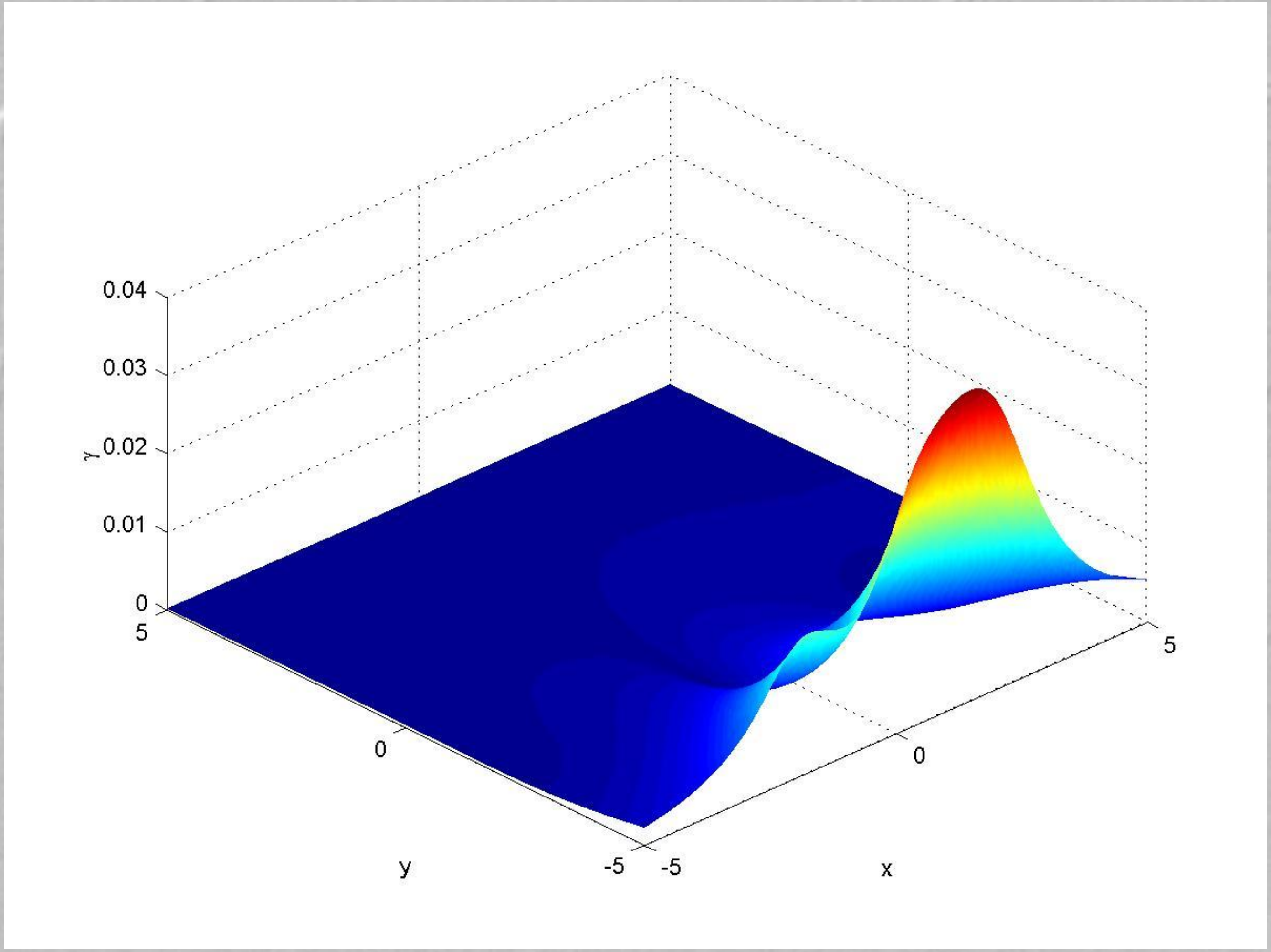
Nelder-Mead Multidimensional Simplex

- Any optimization method may be used and the numerical efficiency of the synthesis algorithm will depend on the method employed.
- We have selected the Nelder-Mead Downhill Simplex Method in Multidimensions.
- Nelder-Mead only requires function evaluations, not derivatives.
- It is relatively inefficient in terms of the required evaluations, but for this problem the computational burden is small.
- Convergence properties are irrelevant since any optimization may be used in the synthesis algorithm.
- The output of the minimization are the values of x and y that minimize the γ of **C**.

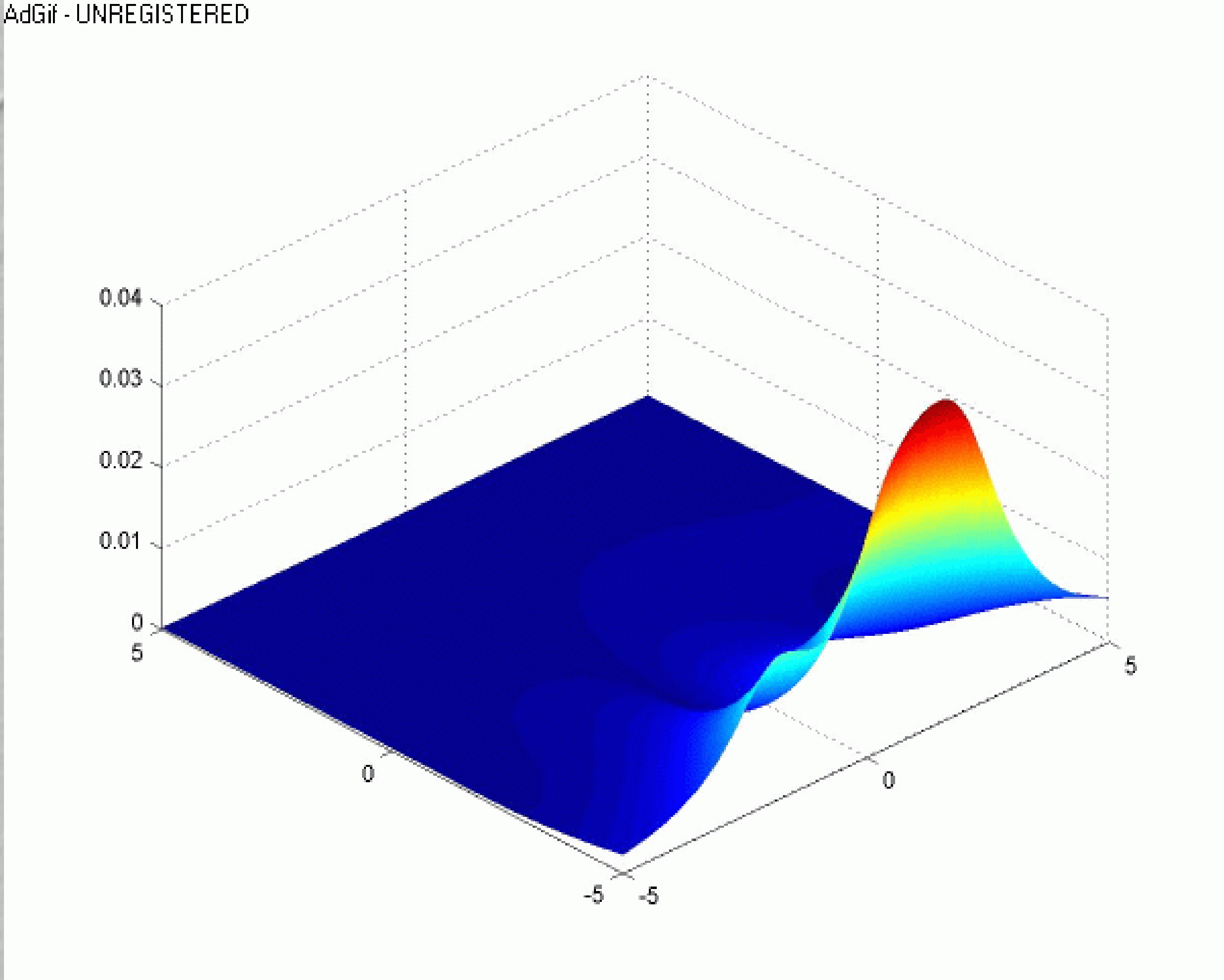
Nelder-Mead Multidimensional Simplex

- The Nelder-Mead algorithm requires an initial guess for x and y.
- We plot γ in terms of x and y in the area of $(x,y) = (0,0)$ up to the maximum distance the coupler attachments are permitted to be relative to moving coupler frame E.
- Within the corresponding parameter space, the approximate local minima are located.
- The two pairs of (x,y) corresponding to the approximate local minimum values of γ are used as initial guesses.
- The Nelder-Mead algorithm converges to the pair of (x,y) coupler attachment point locations that minimize γ within the region of interest.

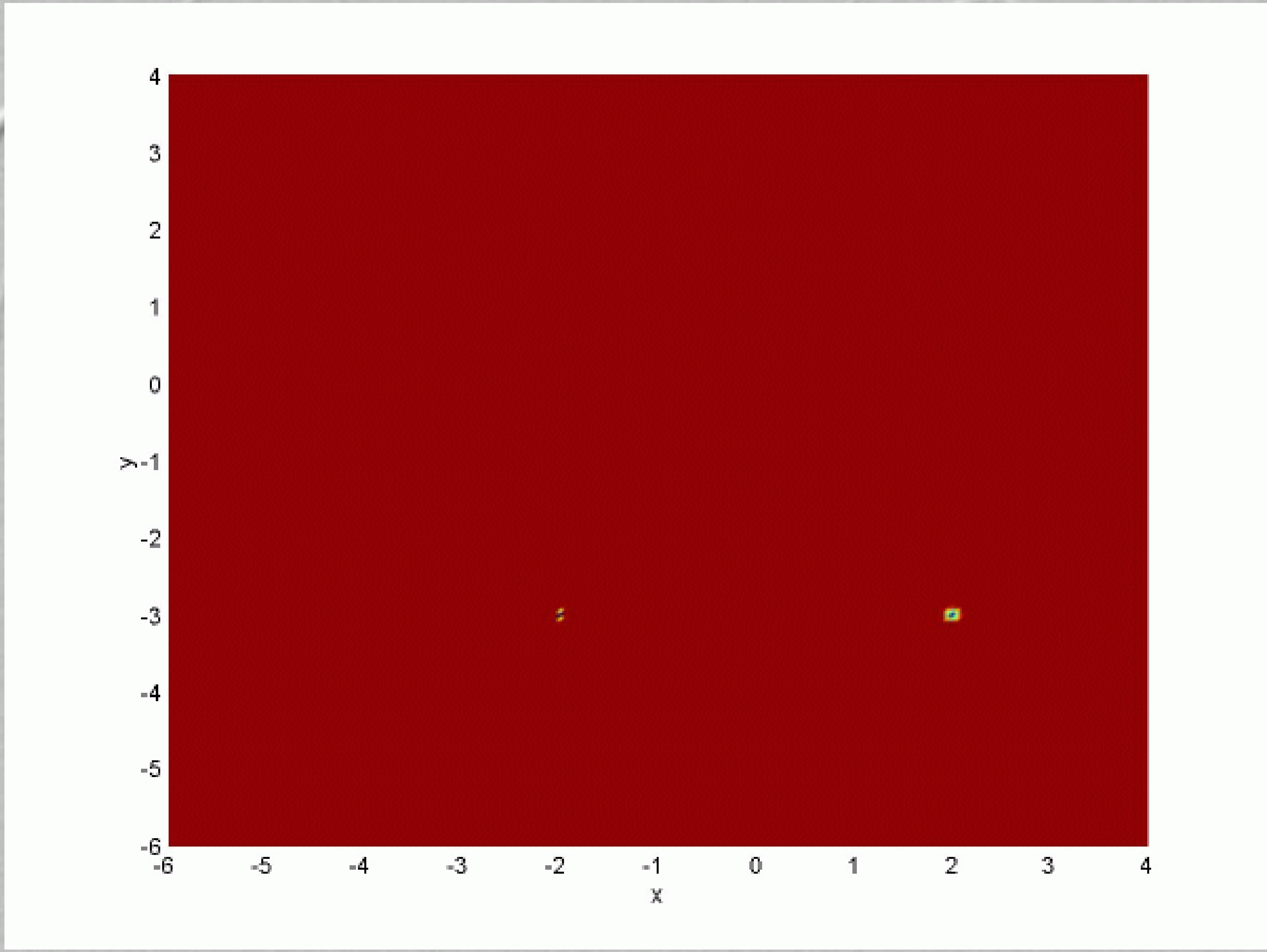
γ Plot



γ Plot



γ Plot



Nelder-Mead Minimization

- Once approximate minima are found graphically, they are input as initial guesses into the Nelder-Mead polytope algorithm
- The output of the minimization is the value of x and y that minimize the γ of **C**

Singular Value Decomposition

Any m x n matrix can be decomposed into:

$$\mathrm{C}_{m\times n} = \mathrm{U}_{m\times m}\mathrm{S}_{m\times n}\mathrm{V}_{n\times n}^{T}$$

where:

- **U** spans the range of **C**
- **V** spans the nullspace of **C**
- **S** contains the singular values of **C**

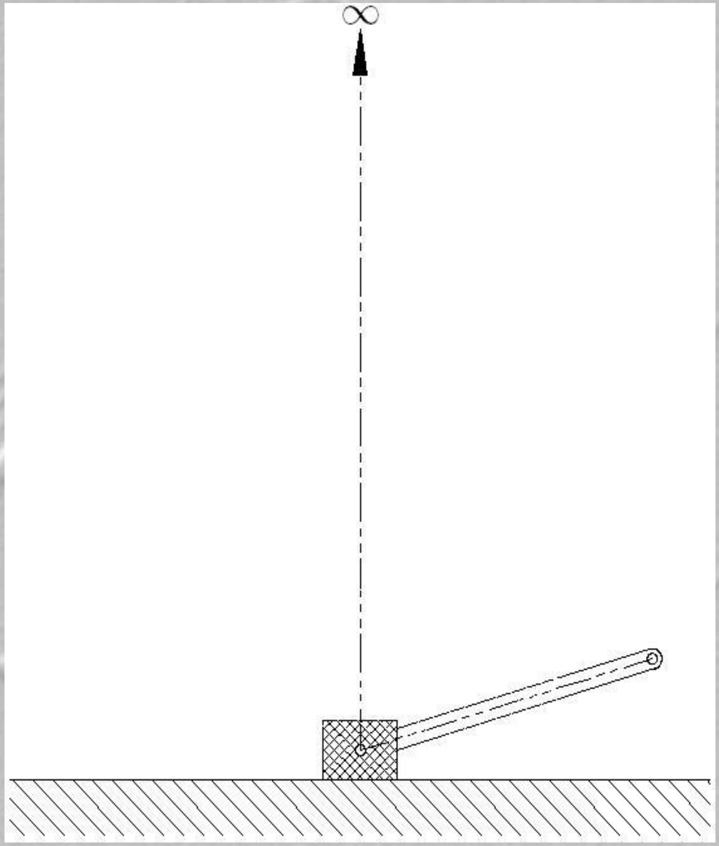
For **C** ill-conditioned (γ minimized):

- The last singular value in **S** is approximately zero
- The last column of **V** is the approximate solution to **CK** = 0

The last column of **V** is then the solution to vector **K**, defining a circle or line

Circle or Line?

- In the most general case, the vector *K* defines a circle, corresponding to an RR dyad
- If the determined circle has dimensions several orders of magnitude greater than the range of the poses, the geometry is recalculated as a line, corresponding to a PR dyad



A PR dyad, analogous to an RR dyad with infinite link length and centered at infinity

Special Cases: The RP Dyad

- RP dyads are the kinematic inverses of PR dyads
- To solve:
  - switch the roles of fixed frame Σ and moving frame *E*
  - Express points *x* and *y* in terms of *X*, *Y*, and θ
  - Solve for constant coordinates (*X*,*Y*) that minimize γ of **C**

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & -b\sin\theta - a\cos\theta \\ -\sin\theta & \cos\theta & b\cos\theta + a\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

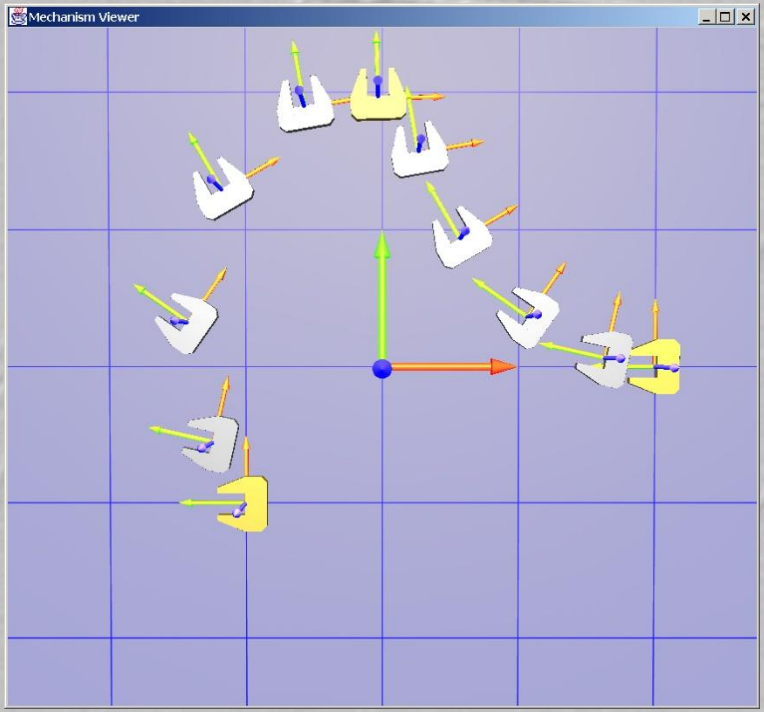
Special Cases: The PP Dyad

PP dyads:

- can *only* produce rectilinear motion at a constant orientation
- can produce *any* rectilinear motion at constant orientation
- are designed based on the practical constraints of the application

Examples: The McCarthy Design Challenge

- Issued at the ASME DETC Conference in 2002
- No information given on the mechanism used to the generate poses
- 11 poses: overconstrained problem



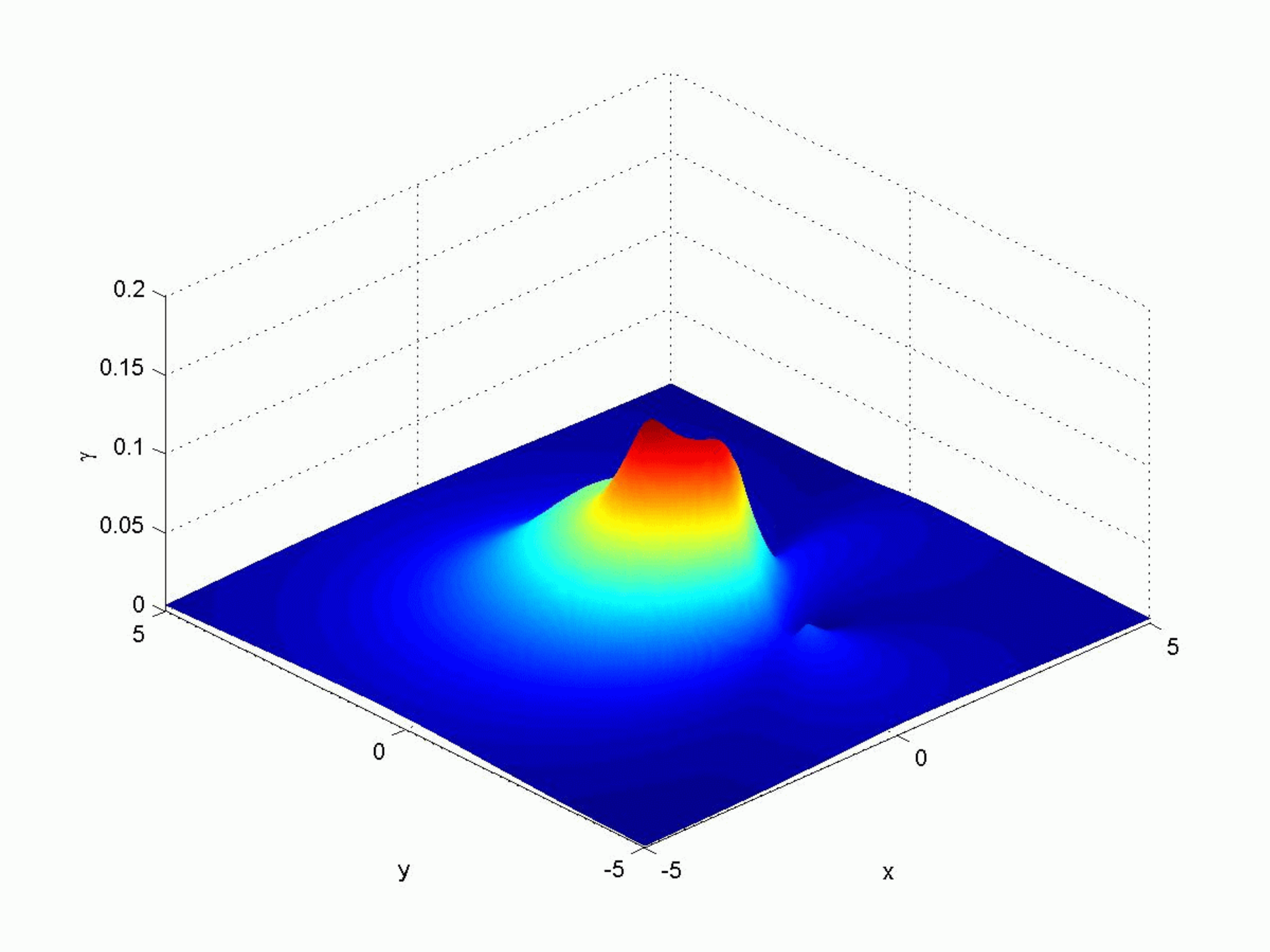
Manufacture Synthesis Matrix

- Substitute pose information into

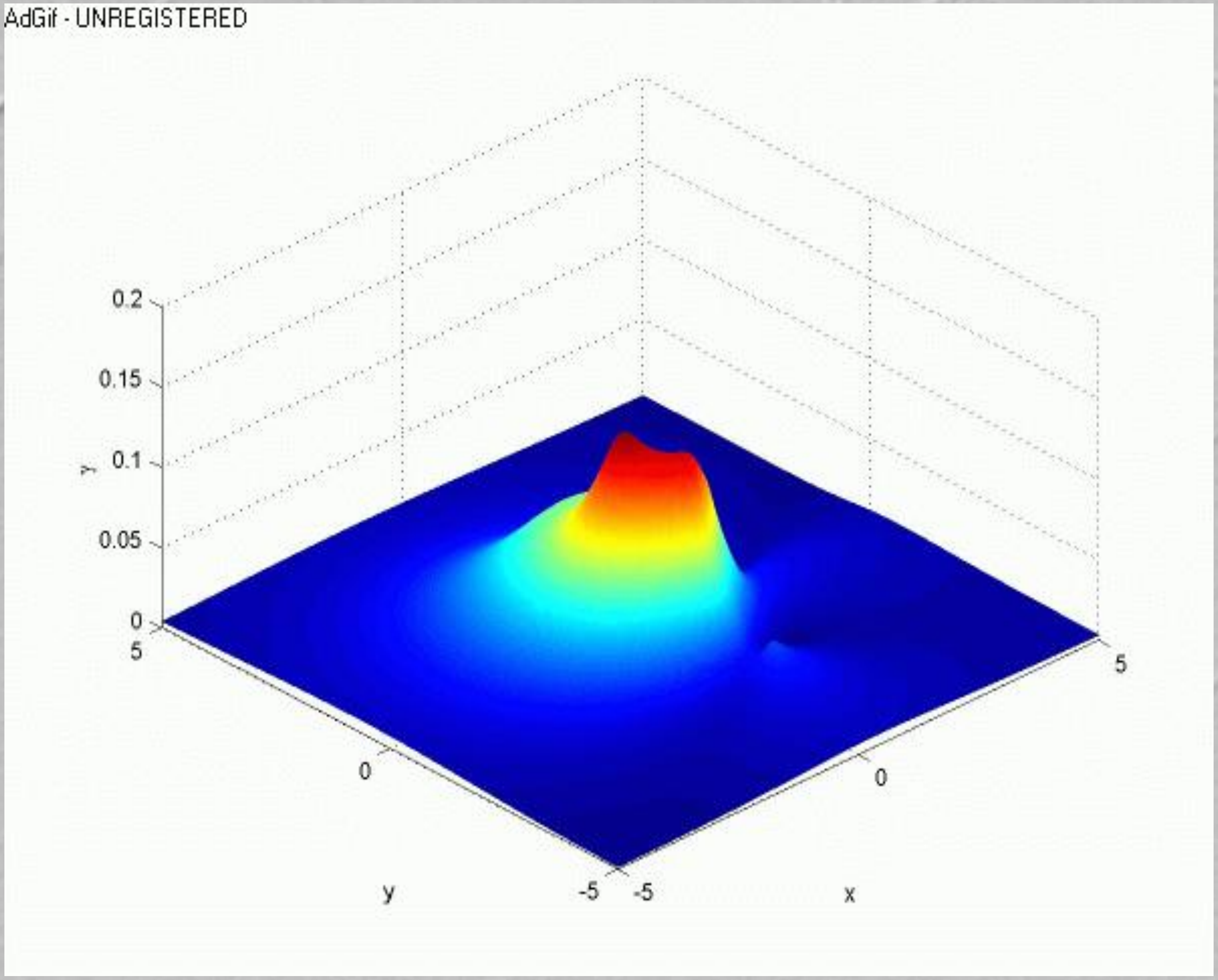
$$\mathbf{CK} = \begin{bmatrix} \left[(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \right]^T \\ \left[2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \right] \\ \left[2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \right] \\ [1] \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = [\mathbf{0}]_{n\times 1}$$

- Plot γ in terms of x and y

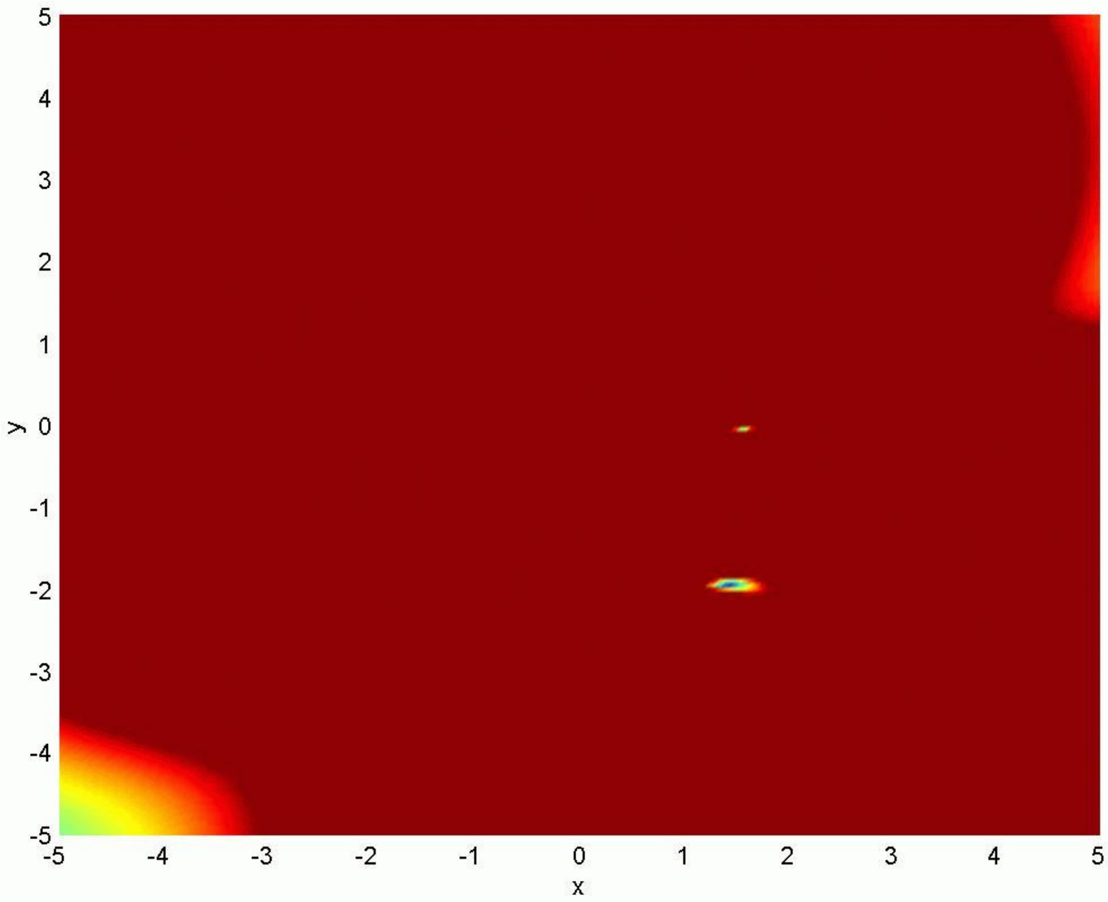
γ Plot



γ Plot



γ Plot



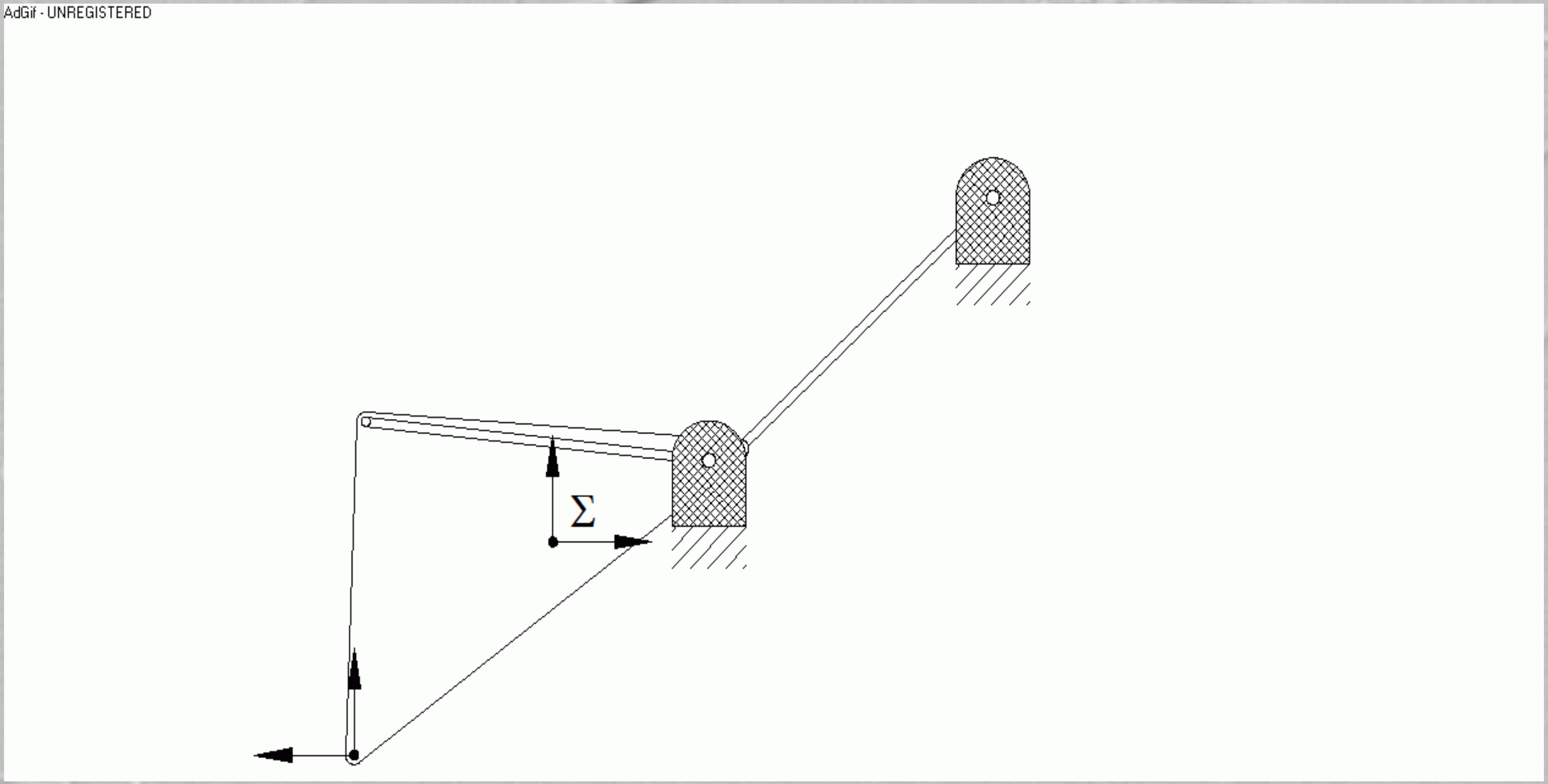
Extracting Mechanism Parameters

- Minima found graphically at approximately (1.5, 0.6), and (1.4, -2.0)
- Using these values as input, Nelder-Mead minimization finds the minima at (1.5656, 0.0583) and (1.4371, -1.9415)
- Singular value decomposition is used to find the *K* vector corresponding to these coordinates

Results

	Dyad 1	Dyad 2
x	1.5656	1.4371
y	-0.0583	-1.9415
K_0	1.0000	1.0000
K_1	-0.7860	-2.2153
K_2	-0.3826	-1.6159
K_3	-2.2390	4.5236

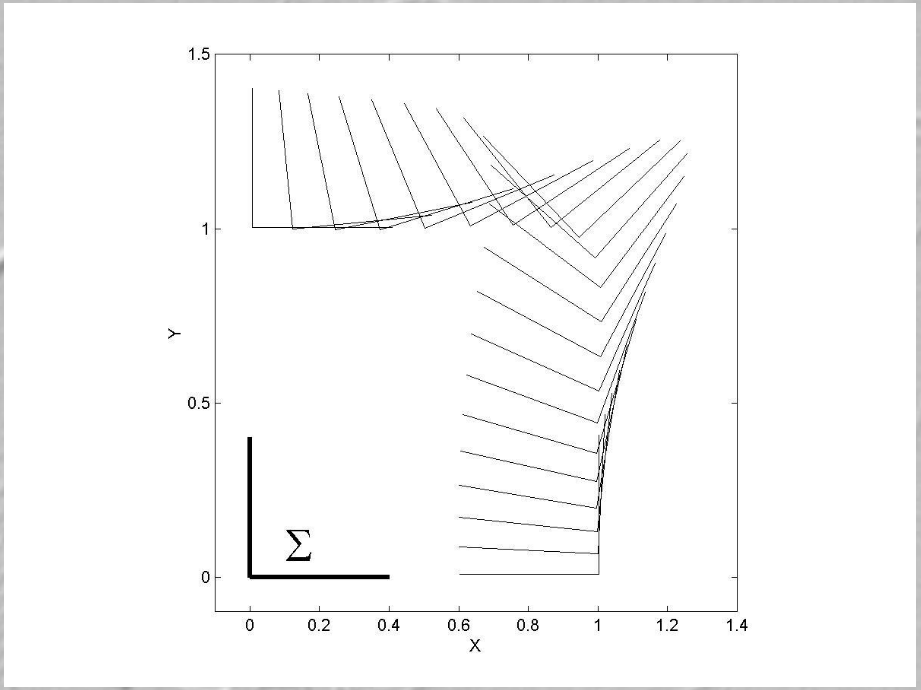
The Solution



Examples: The Square Corner

Exact synthesis is impossible for planar four-bar:

- A PPPP mechanism can replicate the positions, but not the orientations
- The coupler curve of a planar four-bar is at most 6, while a square corner requires infinite order



- Motion from (0,1) to (1,1) to (1,0)
- Orientation decreases linearly from 90 to 0 degrees

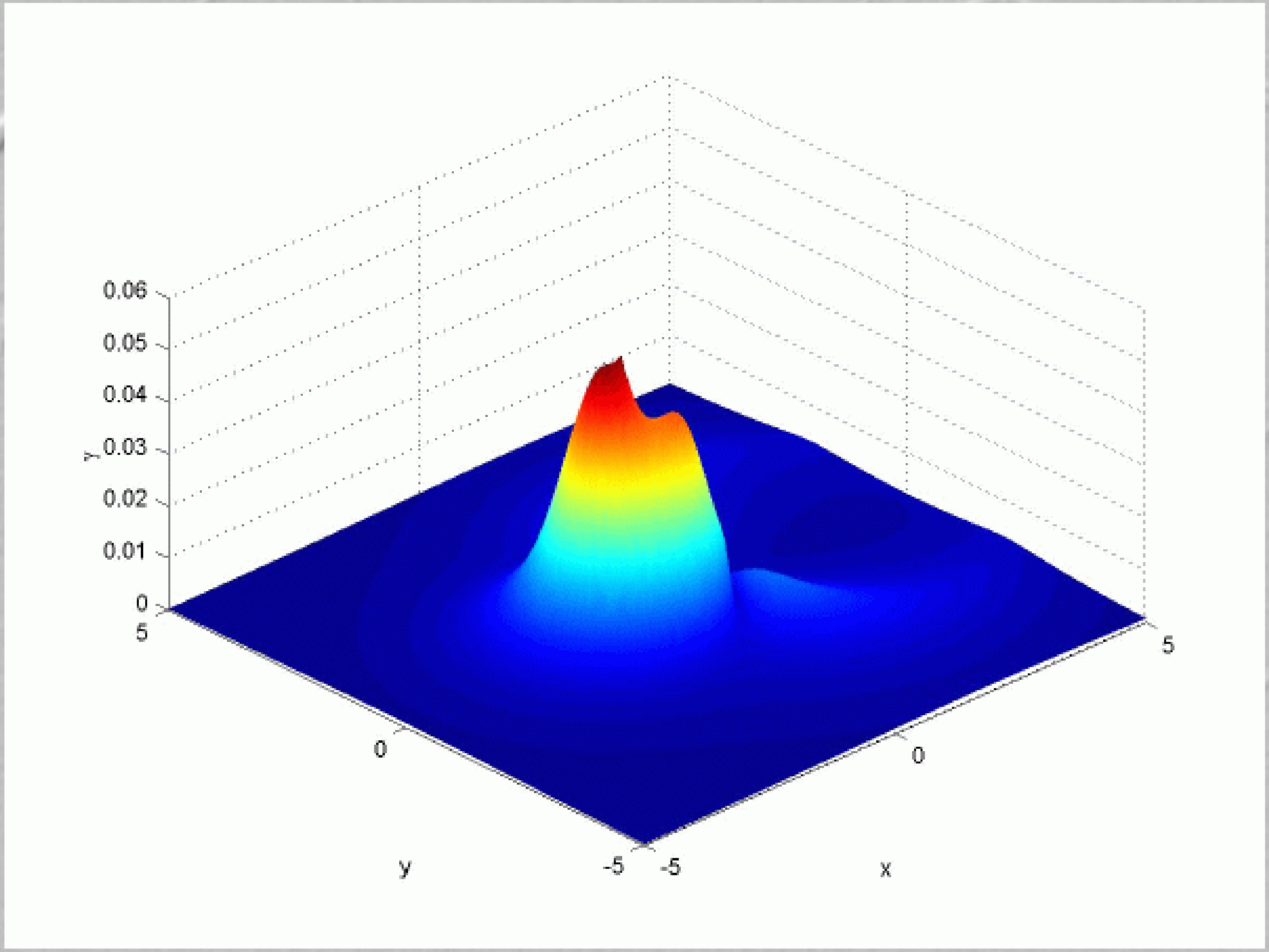
Manufacture Synthesis Matrix

- Substitute pose information into

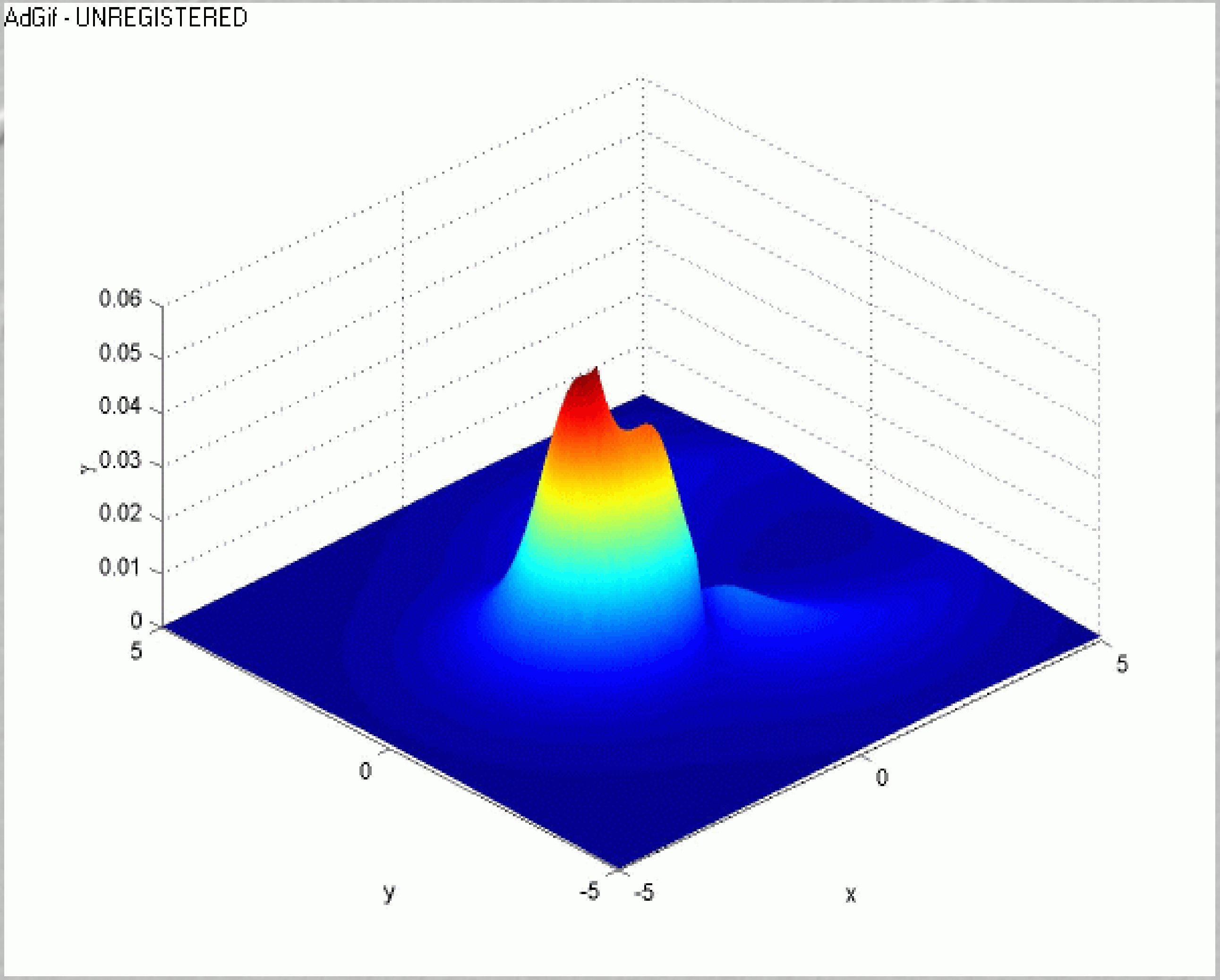
$$\mathbf{CK} = \begin{bmatrix} \left[(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \right]^T \\ \left[2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \right] \\ \left[2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \right] \\ [1] \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = [\mathbf{0}]_{n\times 1}$$

- Plot γ in terms of x and y

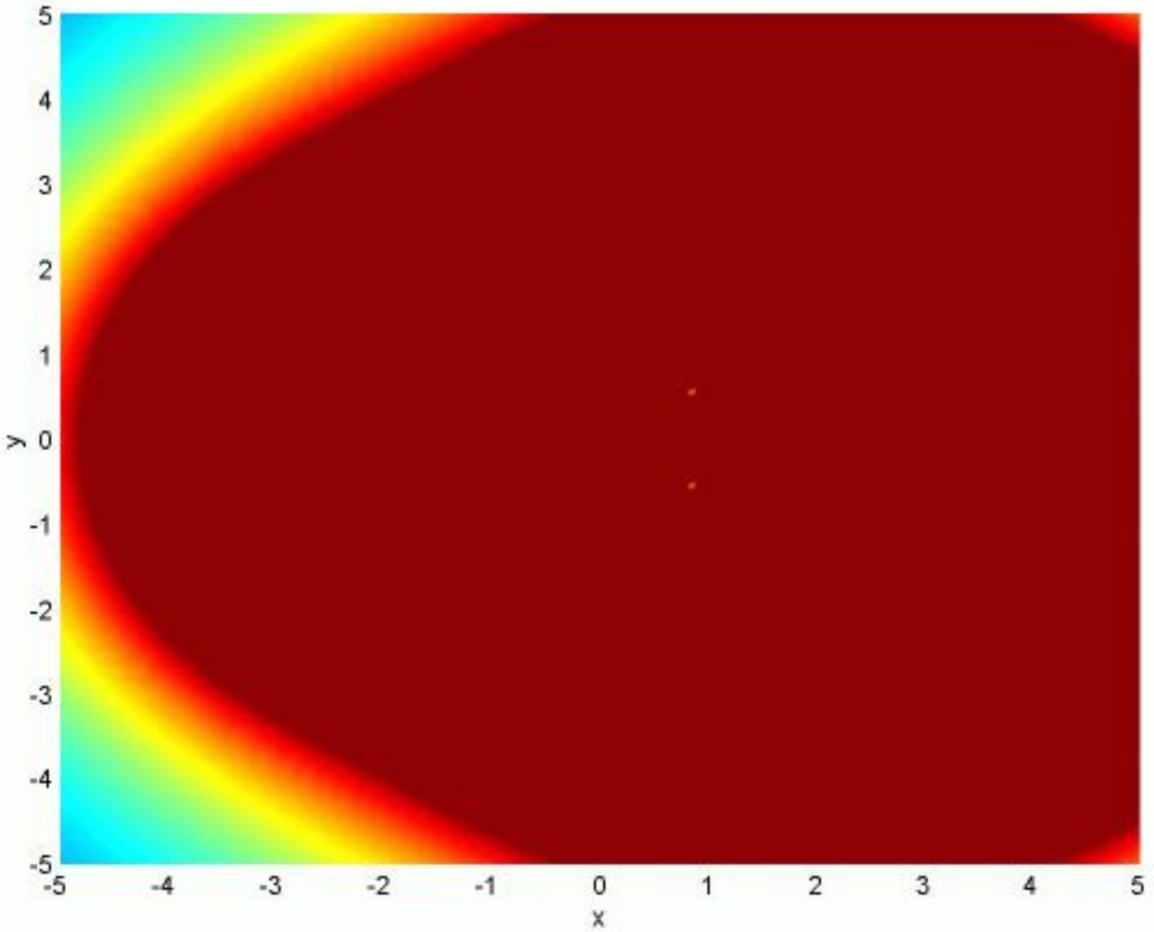
γ Plot



γ Plot



γ Plot



Extracting Mechanism Parameters

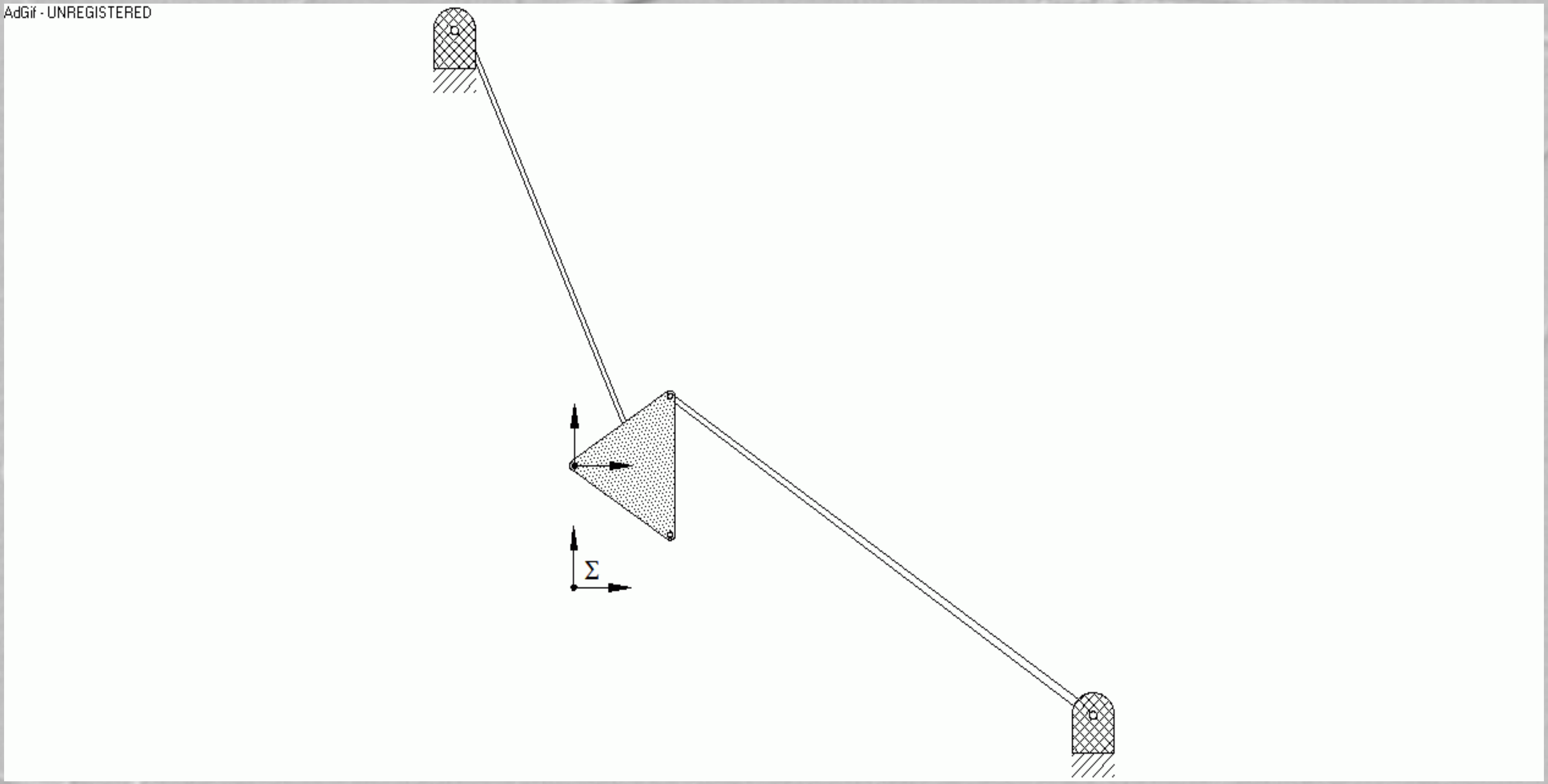
- Minima found graphically at approximately (0.8,0.6), and (0.8,-0.6)
- Using these values as input, Nelder-Mead minimization finds the minima at

  (0.8413,0.5706) and (0.8413,-0.5706)
- Singular value decomposition is used to find the *K* vector corresponding to these coordinates

Results

	Dyad 1	Dyad 2
x	0.8413	0.8413
y	0.5706	-0.5706
K_0	1.0000	1.0000
K_1	-4.5843	1.0539
K_2	1.0539	-4.5843
K_3	1.2704	1.2704

The Solution



Conclusions

- This method determines type and dimensions of mechanisms that best approximate $n > 5$ poses in a least squares sense
- No initial guess is necessary
- Examples illustrate utility and robustness

Quadric Surface Fitting Applications to Approximate Dimensional Synthesis

© M.J.D. Hayes, S. Radacina-Rusu

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Proceedings of the 13th IFToMM World Congress
June 19-23, 2011
Guanajuato, México

Fitting Image Space Points (Displacements) to Constraint Surfaces

- Given a suitably over constrained set of image space coordinates X_1, X_2, X_3, and X_4 which represent the desired set of positions and orientations of the coupler identify the constraint surface shape coefficients: K_0, K_1, K_2, K_3, x, and y.
- The given image space points are on some space curve.
- Project these points onto the *best* 4^{th} order curve of intersection of two quadric constraint surfaces.
- These intersecting surfaces represent two dyads in a mechanism that possesses displacement characteristics closest to the set of specified poses.

Identifying the Constraint Surfaces

- Surface type is embedded in in the coefficients of its implicit equation:

$$c_0 X_4^2 + c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_1 X_2 + c_5 X_2 X_3 +$$
$$c_6 X_3 X_1 + c_7 X_1 X_4 + c_8 X_2 X_4 + c_9 X_3 X_4 = 0.$$

- It can be classified according to certain invariants of its discriminant and quadratic forms:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}^T \begin{bmatrix} c_1 & \frac{1}{2}c_4 & \frac{1}{2}c_6 & \frac{1}{2}c_7 \\ \frac{1}{2}c_4 & c_2 & \frac{1}{2}c_5 & \frac{1}{2}c_8 \\ \frac{1}{2}c_6 & \frac{1}{2}c_5 & c_3 & \frac{1}{2}c_9 \\ \frac{1}{2}c_7 & \frac{1}{2}c_8 & \frac{1}{2}c_9 & c_0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \mathbf{X}^T \mathbf{D} \mathbf{X} = 0.$$

Identifying the Constraint Surfaces

- Given a sufficiently large number *n* of poses expressed as image space coordinates yields *n* equations linear in the c_i coefficients

$$c_0 X_4^2 + c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_1 X_2 + c_5 X_2 X_3 +$$
$$c_6 X_3 X_1 + c_7 X_1 X_4 + c_8 X_2 X_4 + c_9 X_3 X_4 = 0.$$

- The n equations can be re-expressed as:

$$\mathbf{Ac} = \mathbf{0}.$$

- The same numbered elements in Matrix **A**, corresponding to the X_i are scaled by the unknown c_i.

Identifying the Constraint Surfaces

- Applying SVD to Matrix **A** reveals the vectors **c** that are in, or computationally close, in a least-squares sense, to the nullspace of **A**.
- Certain invariants of the resulting discriminant and corresponding quadratic form reveal the nature of the quadric surface.
- *RR* dyads require the quadric surface to be an hyperboloid of one sheet with certain properties.
- *RP* and *PR* dyads require the quadric surface to be an hyperbolic paraboloid.

Equivalent Minimization Problem

- Assuming the mechanism type has been identified given $n >> 5$ specified poses, the approximate synthesis problem can be solved using an equivalent unconstrained non-linear minimization problem.
- It can be stated as “find the surface shape parameters that minimize the total spacing between all points on the specified reference curve and the same number of points on a dyad constraint surface”.
- First the constraint surfaces are projected into the space corresponding to the hyperplane $X_4=1$.
- This yields the following parameterizations:

Equivalent Minimization Problem

- Hyperboloid of one sheet:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} ([x - K_1]t + K_2 + y) + (r\sqrt{t^2+1})\cos\gamma \\ ([y - K_2]t - K_1 - x) + (r\sqrt{t^2+1})\sin\gamma \\ 2t \end{bmatrix},$$

$$\gamma \in \{0, \cdots, 2\pi\},\ t \in \{-\infty, \cdots, \infty\},$$

x and y are the coordinates of the moving revolute centre expressed in the moving coordinate system E,

K_1 and K_2 are the coordinates of the fixed revolute centre expressed in the fixed coordinate system Σ,

r is the distance between fixed and moving revolute centres,

while *t* and *γ* are free parameters.

Equivalent Minimization Problem

- Hyperbolic paraboloid:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \\ t \end{bmatrix} + s \begin{bmatrix} -b(t) \\ a(t) \\ 0 \end{bmatrix}$$

$$t \in \{-\infty, \cdots, \infty\},\ s \in \{-\infty, \cdots, \infty\},$$

  $f(t)$, $g(t)$, $a(t)$, and $b(t)$ are functions of the surface shape parameters and the free parameter t,

  while s is another free parameter.

- Note that in both cases the X_3 coordinate varies linearly with the free parameter t, and can be considered another free parameter.

Equivalent Minimization Problem

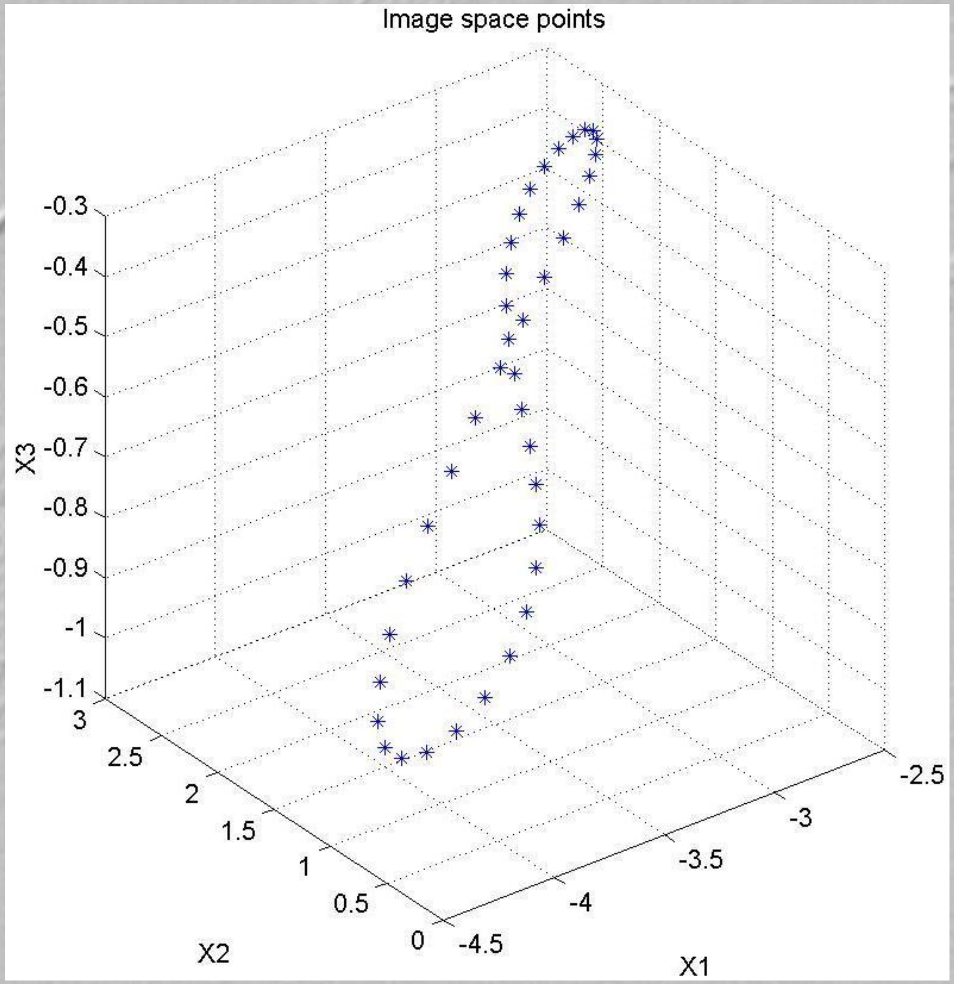
- The total distance between the specified reference image space points on the reference curve and corresponding points that lie on a constraint surface where $t = X_3 = X_{3_{\text{ref}}}$ is defined as

$$d = \sum_{i=1}^{n} \sqrt{(X_{1_{\text{ref}}} - X_{1_i})^2 + (X_{2_{\text{ref}}} - X_{2_i})^2}$$

- The two sets of surface shape parameters that minimize d represent the two best constraint surfaces that intersect closest to the reference curve.
- The distance between each reference point and each corresponding point on the quadric surface in the hyperplane $t = X_3 = X_{3_{\text{ref}}}$ can measured in the plane spanned by X_1 and X_2.

Example

- A planar 4*R* linkage was used to generate 40 poses of the coupler.
- The resulting image space points lie on the curve of intersection of two hyperboloids of one sheet.
- The reference curve can be visualized in the hyperplane X_4=1.



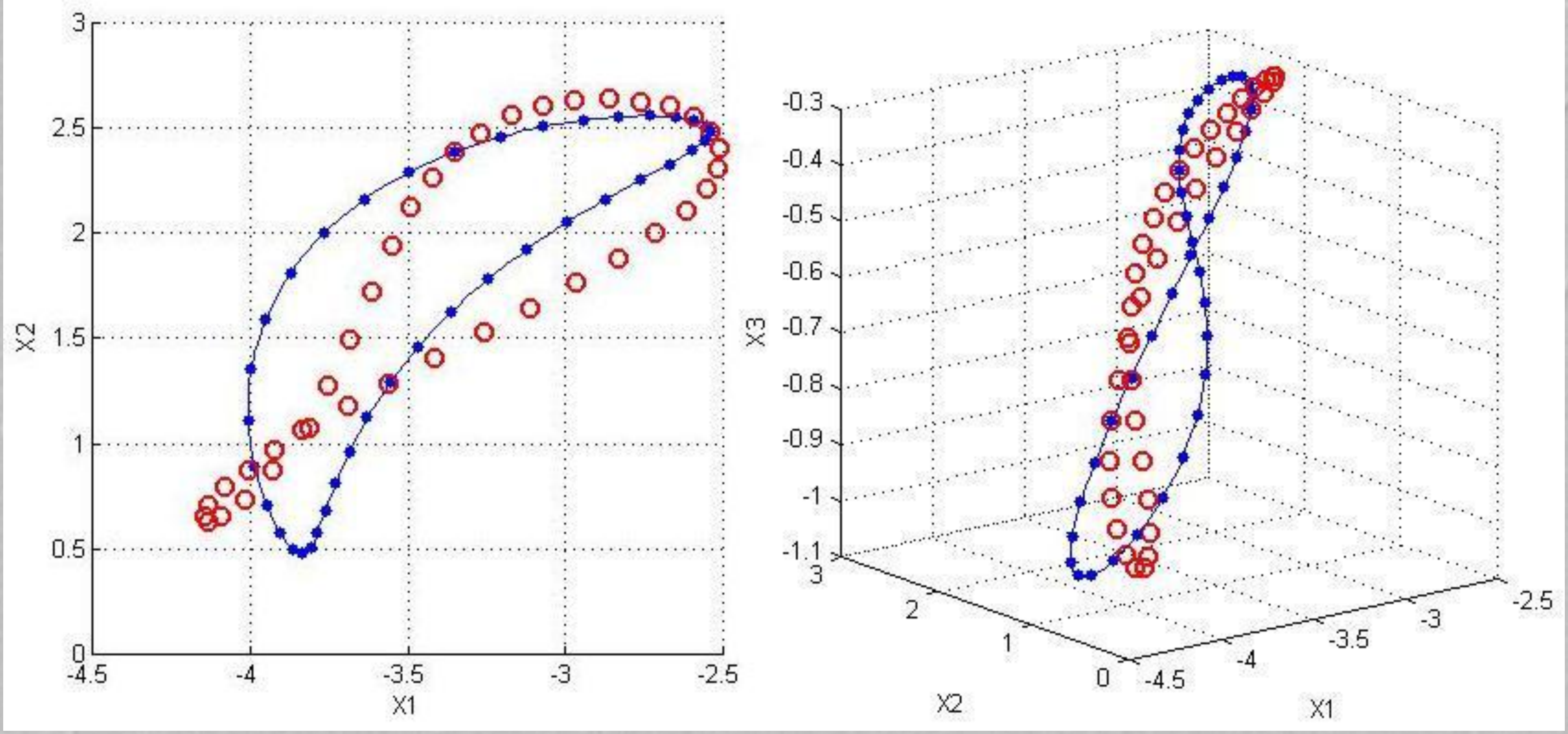
Example

- In order for the algorithm to converge to the solution that minimizes *d*, decent initial guesses for the surface shape parameters are required.
- Out of the 40 reference points, sets of five were arbitrarily chosen spaced relatively wide apart yielding sets of five equations in the five unknown shape parameters.
- Solving yields the initial guesses.
- Non-linear unconstrained algorithms such as the Nelder-Mead simplex and the Hookes-Jeeves methods were used with similar outcomes.

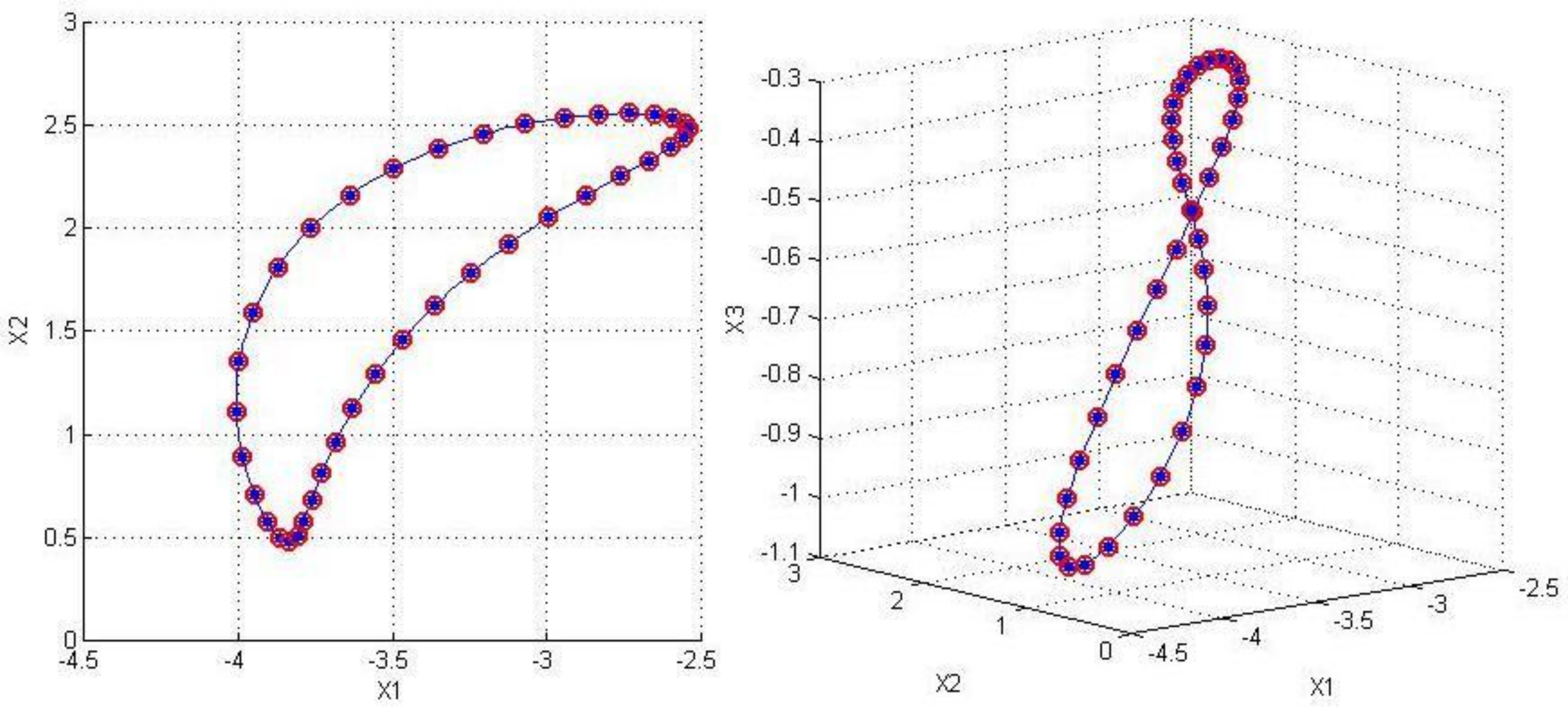
Results Generated From Initial Guesses

Parameter	Guess 1	Guess 2	Guess 3	Guess 4	Guess 5	Guess 6	Guess 7
K_1	-97.720	-18.202	888.914	-5.000	1.000	-25.445	-1.398
K_2	-57.463	-12.363	432.395	0.000	-1.000	-17.073	-6.191
K_3	1491.757	261.650	-2374.375	21.000	-23.000	390.531	36.554
x	-1.133	-1.287	-0.894	3.000	-1.000	-1.309	-4.388
y	0.534	0.889	-5.375	-2.000	-2.000	1.030	-2.361
Iterations	450	623	718	101	176	745	436
d	1.1132	1.9333	6.726	0.0004	0.0010	1.5746	4.8138

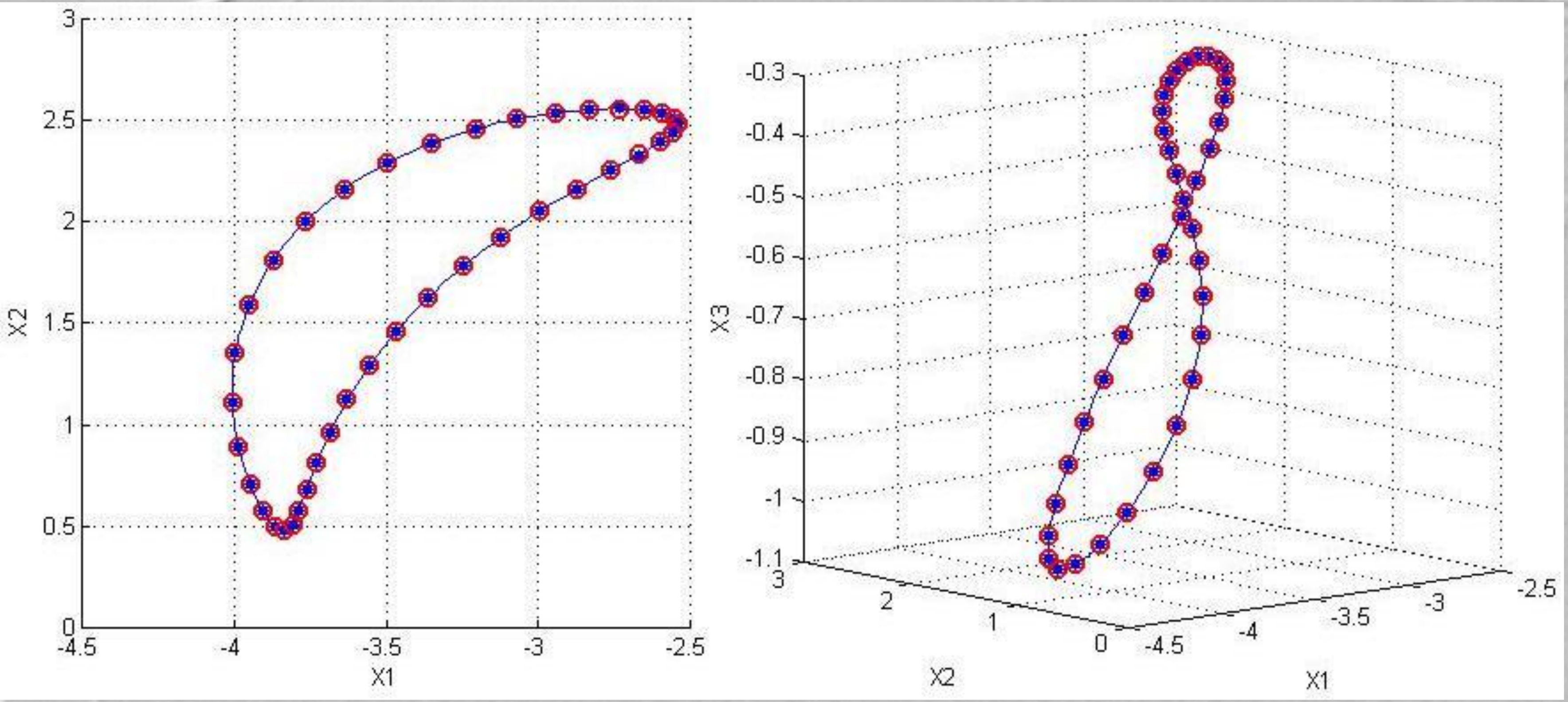
Results: Initial Guess 3



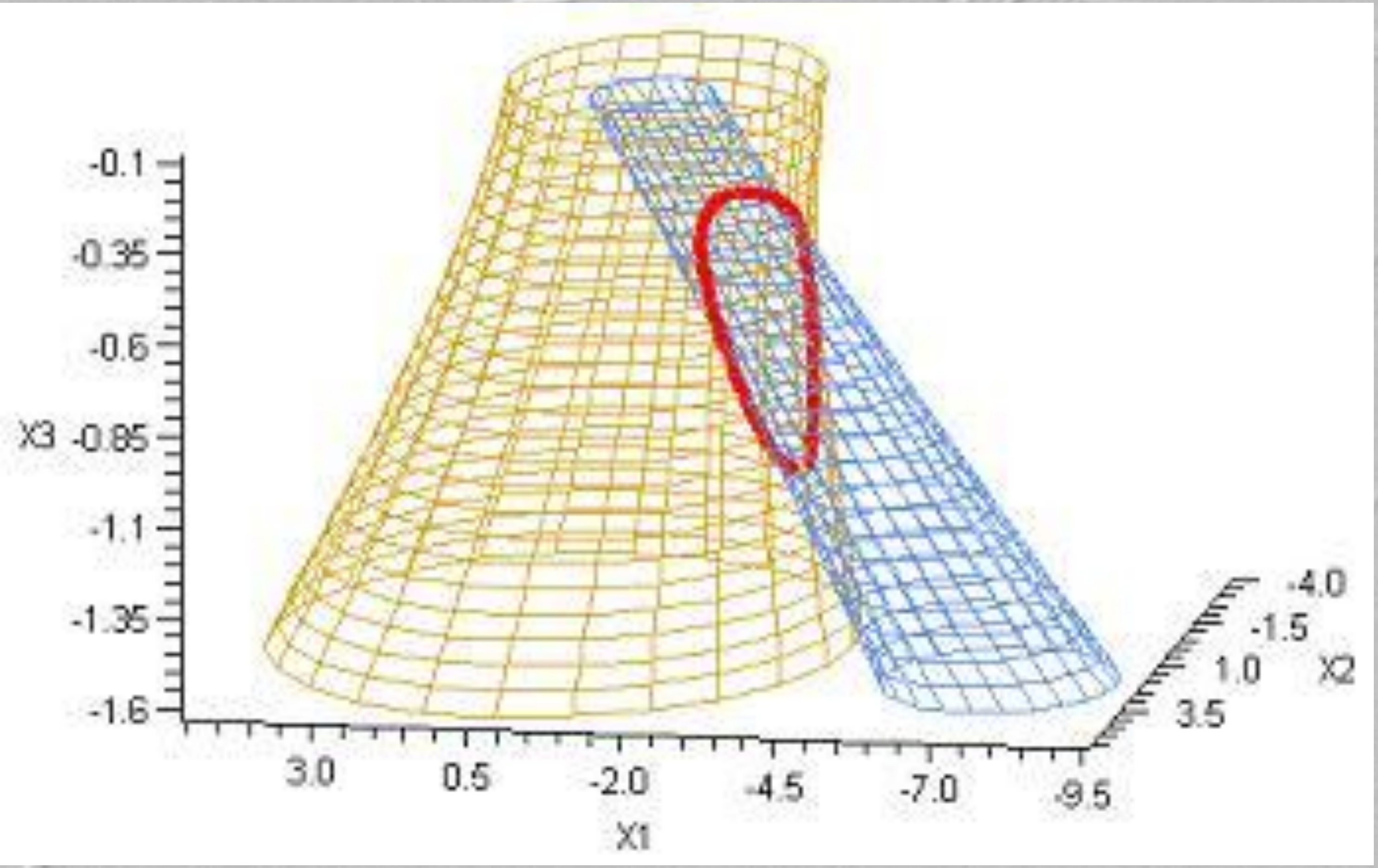
Results: Initial Guess 4



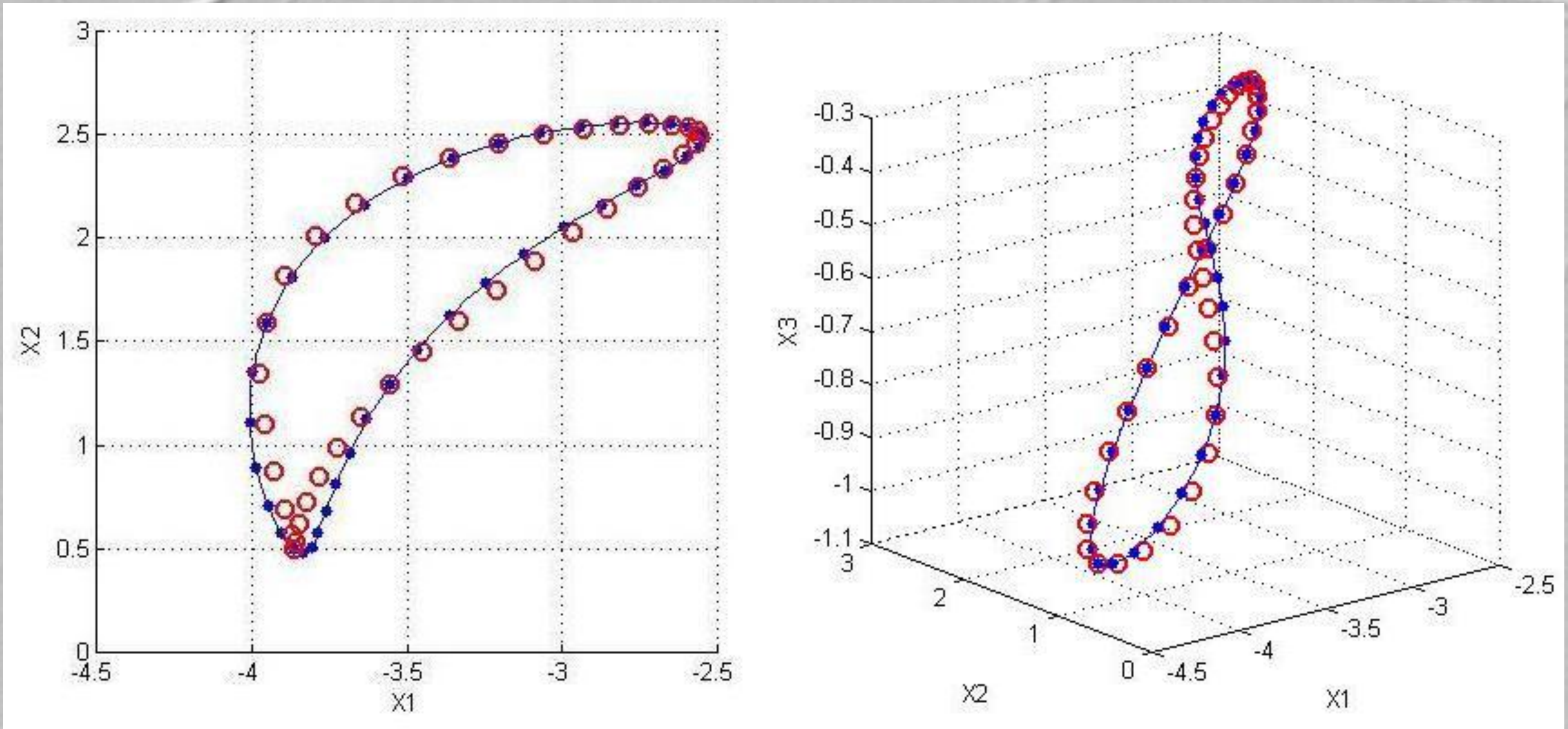
Results: Initial Guess 5



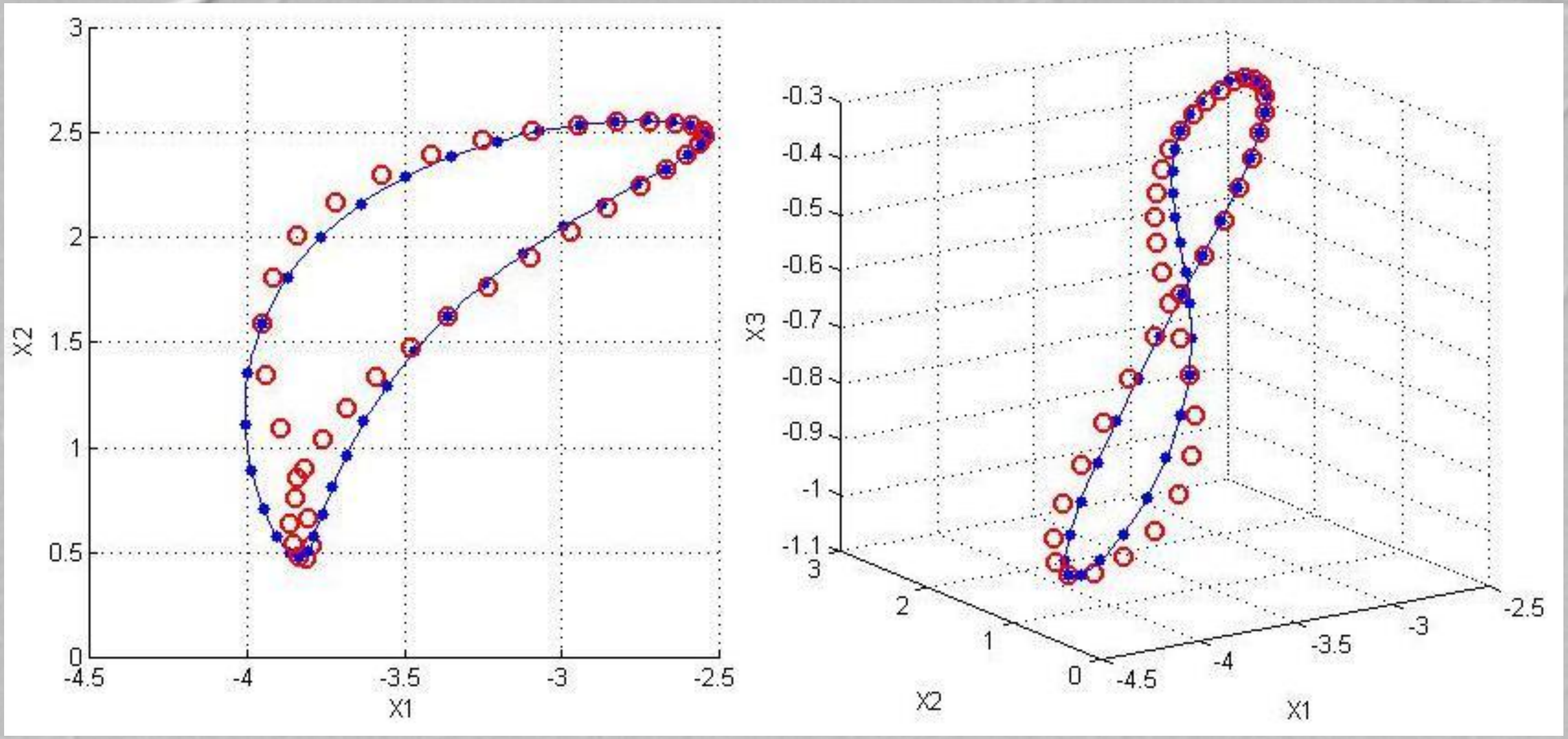
Intersection of Two *Best* Surfaces



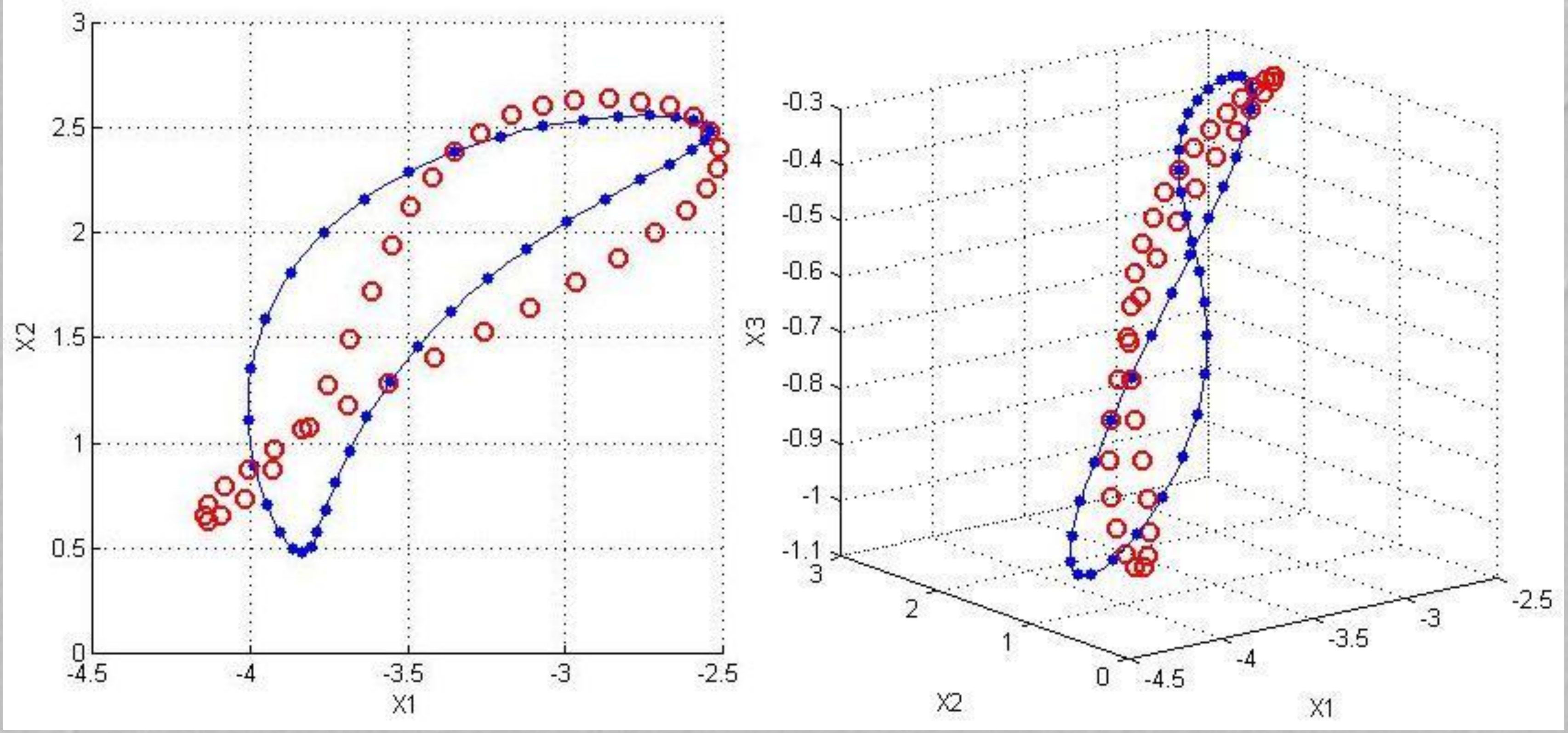
Results: Initial Guess 1



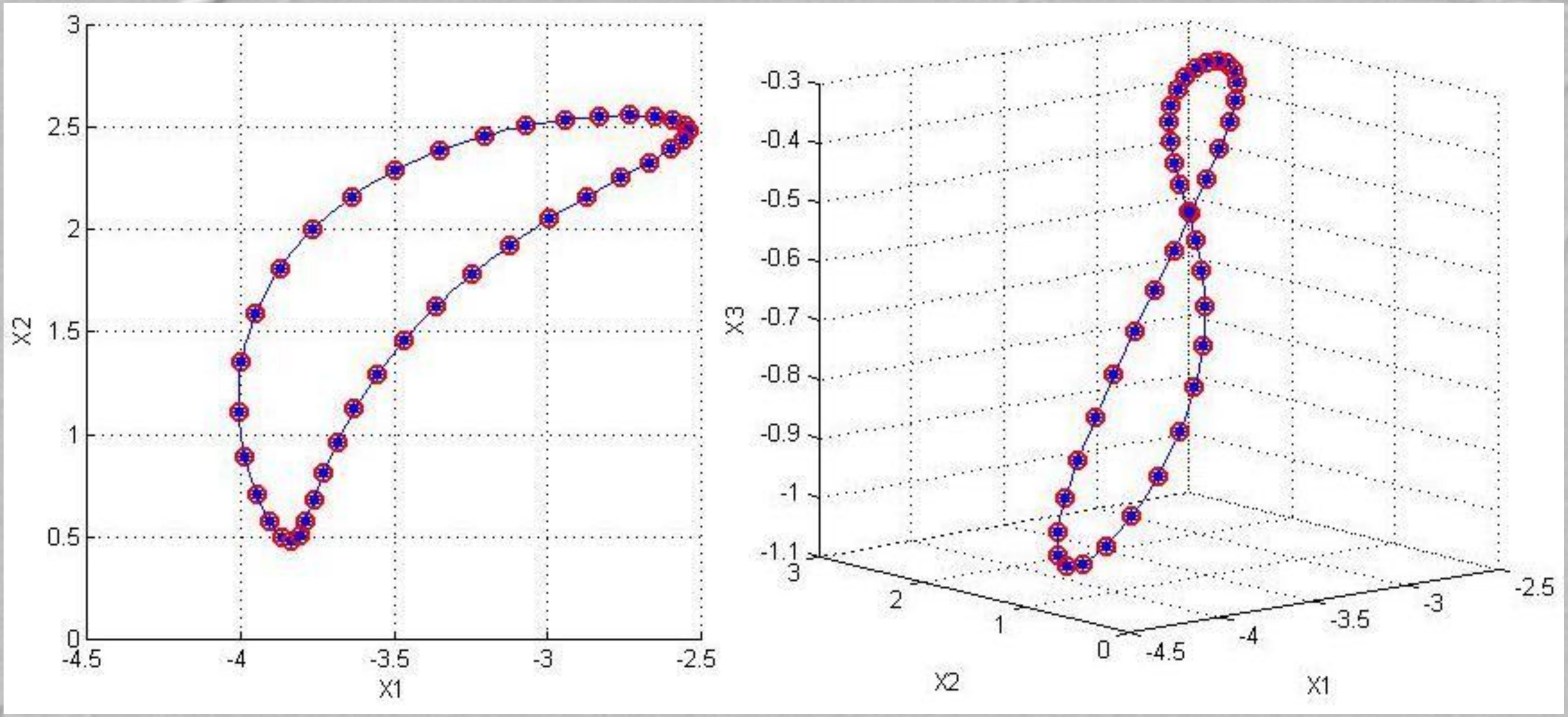
Results: Initial Guess 2



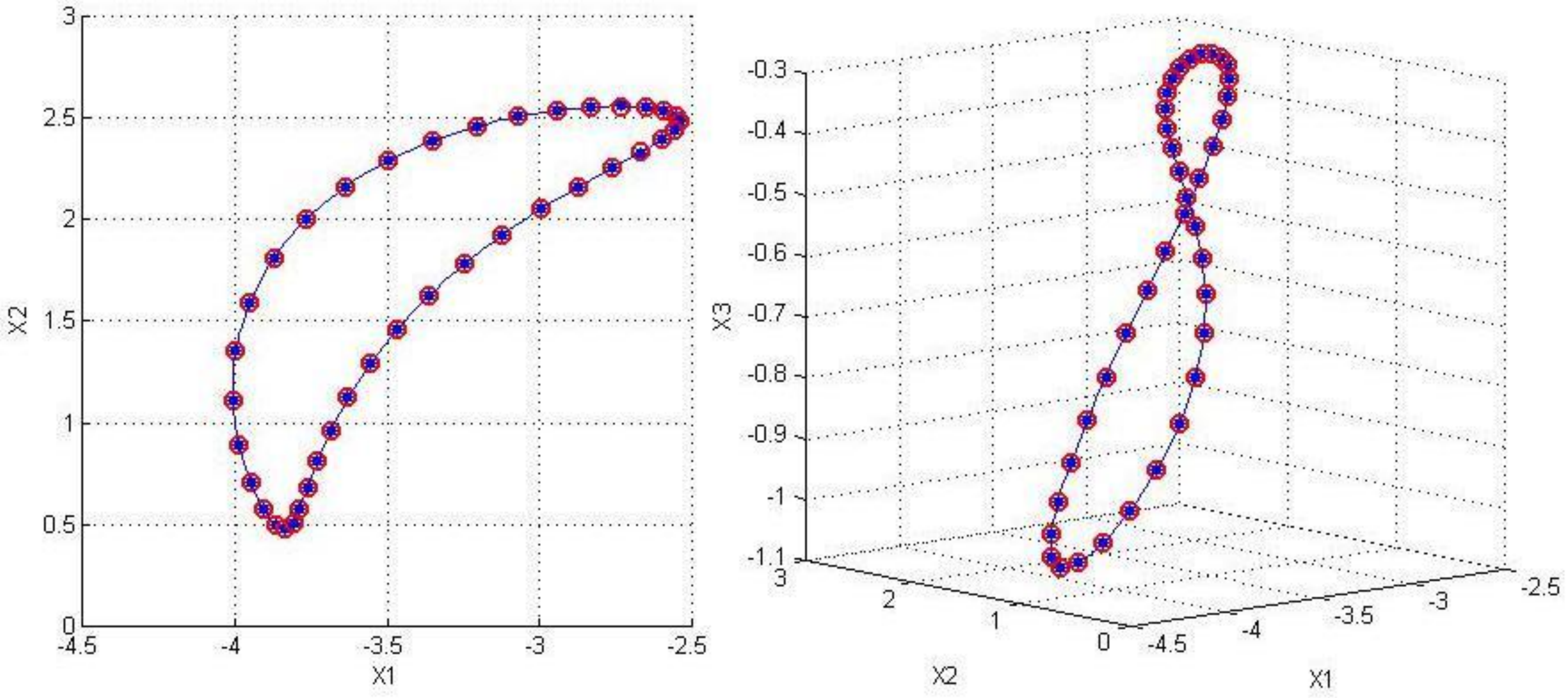
Results: Initial Guess 3



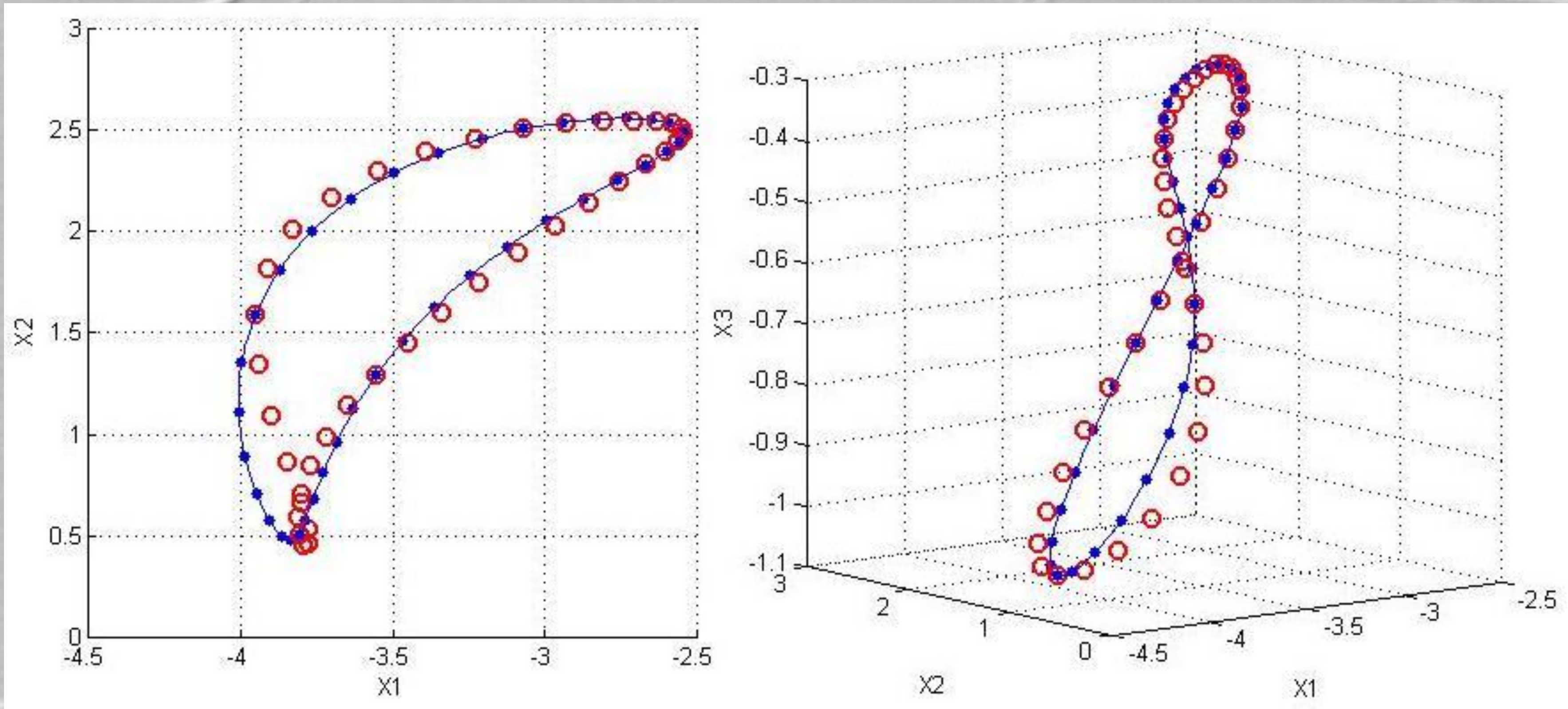
Results: Initial Guess 4



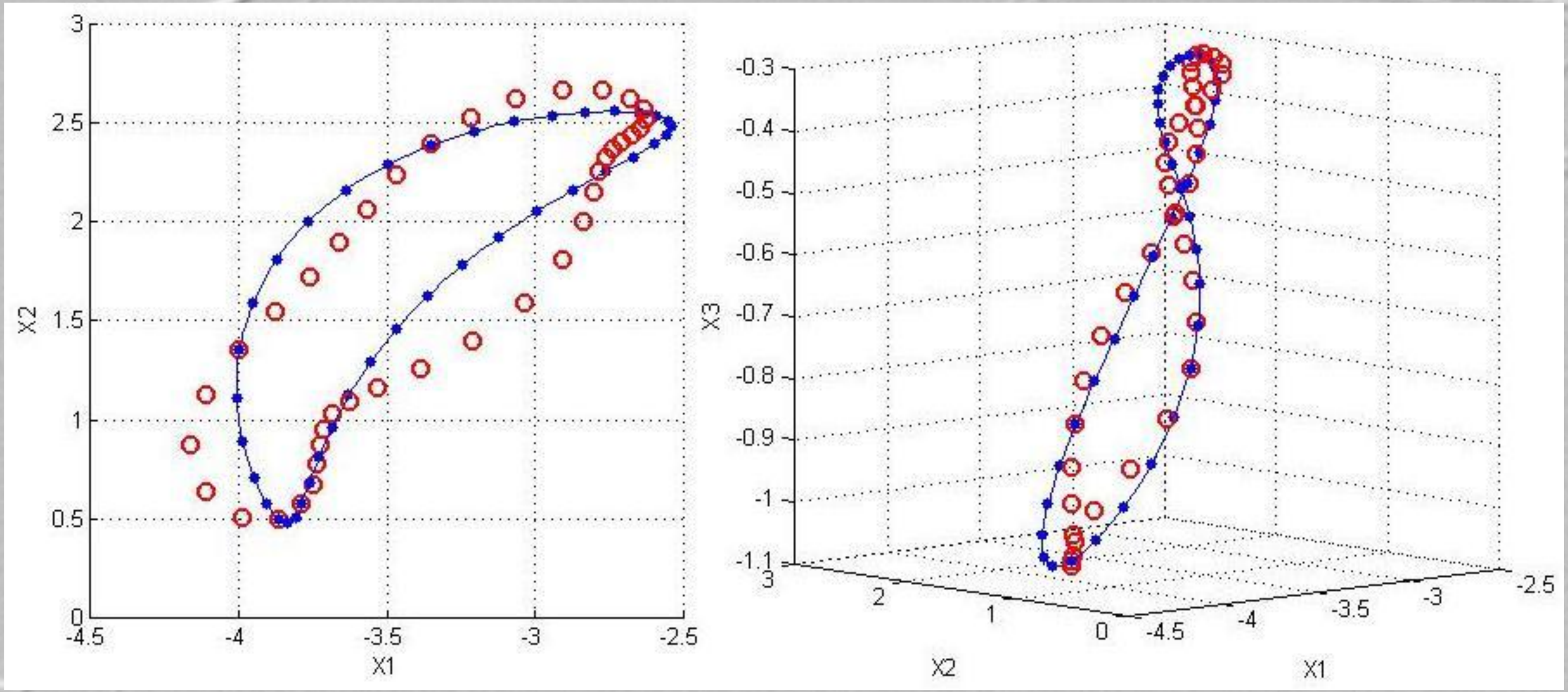
Results: Initial Guess 5



Results: Initial Guess 6



Results: Initial Guess 7



Conclusions

- A new approximate synthesis algorithm was developed minimizing the total deviation, *d*, from specified poses represented as points in the kinematic mapping image space.
- No heuristics are necessary and only five variables are needed.
- The algorithm returns a list of best generating mechanisms ranked according to *d*.
- The minimization could be further developed to jump from local minima to other local minima depending on desired "closeness" to specified poses.
- Relationships between the surface shape parameters may be exploited so the algorithm recognizes undesirable solutions and avoids iterations in those directions.

MECH 5507
Advanced Kinematics

Applications to Analysis

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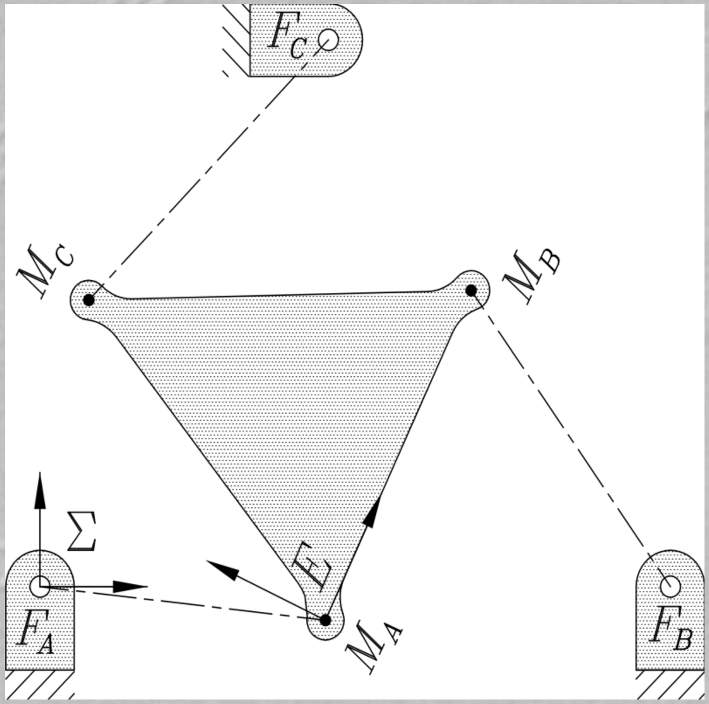
Applications to Analysis

- Kinematic mapping can also be used effectively for the analysis of complex kinematic chains.
- A very common example is a planar three-legged manipulator.
- A moving rigid planar platform connected to a fixed rigid base by three open kinematic chains. Each chain is connected by 3 independent 1 DOF joints, one of which is active.

General Planar Three-Legged Platforms

- 3 arbitrary points in a particular plane, described by frame *E*, that can have constrained motion relative to 3 arbitrary points in another parallel plane, described by frame Σ.
- Each platform point keeps a certain distance from the corresponding base point. These distances are set by the variable joint parameter and the topology of the kinematic chain.

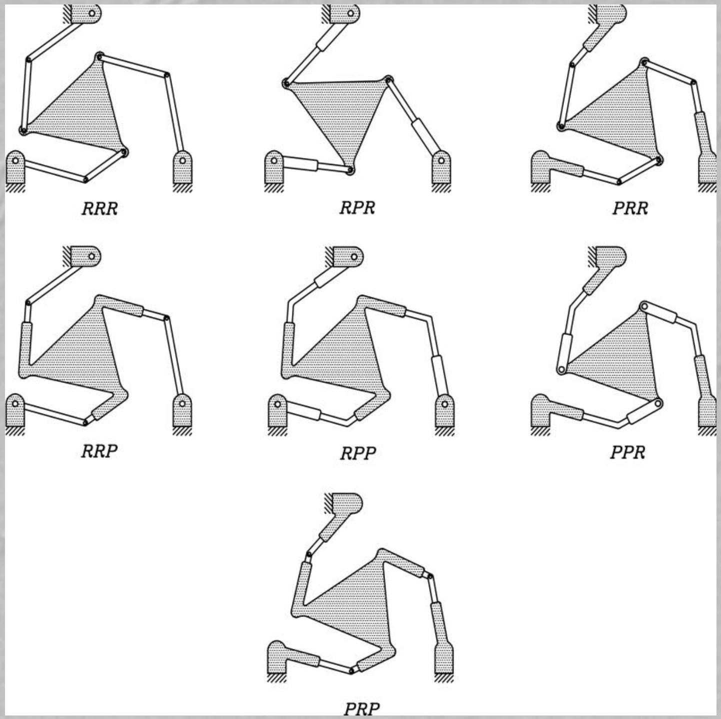


Characteristic Chains

- The possible combinations of *R* and *P* pairs of 3 joints starting from the fixed base are:

  RRR, *RPR*, *RRP*, *RPP*, *PRR*, *PPR*, *PRP*, *PPP*

- The *PPP* chain is excluded since no combination of translations can cause a rotation.
- 7 possible topologies each characterized by one simple chain.



Passive Sub-Chains

- There are 21 possible joint actuation schemes, as any of the 3 joints in any of the 7 characteristic chains may be active.
- When the active joint input is set, the remaining passive sub-chain is one of the following 3:

RR, *PR*, *RP*

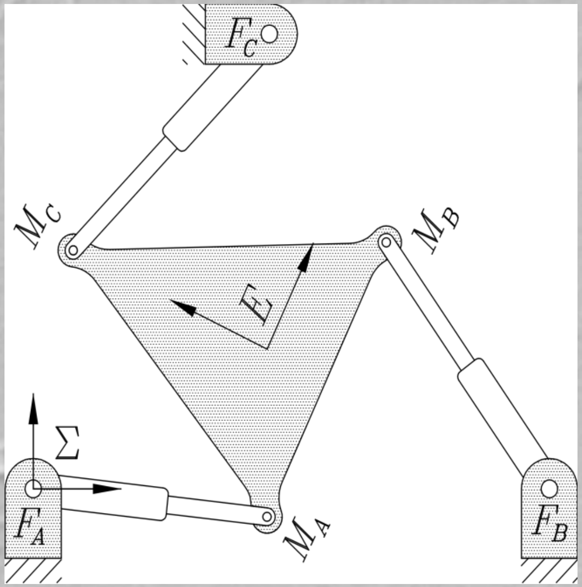
- The *PP*-type sub-chains are disregarded because platforms containing such sub-chains are more likely to be architecture singular.
- Thus, the number of different three-legged platforms is

$$C(n,k) = \frac{(n+k-1)!}{k!(n-1)!} \Rightarrow C(18,3) = 1140$$

- The direct kinematic analysis of all 1140 types is possible with this method.

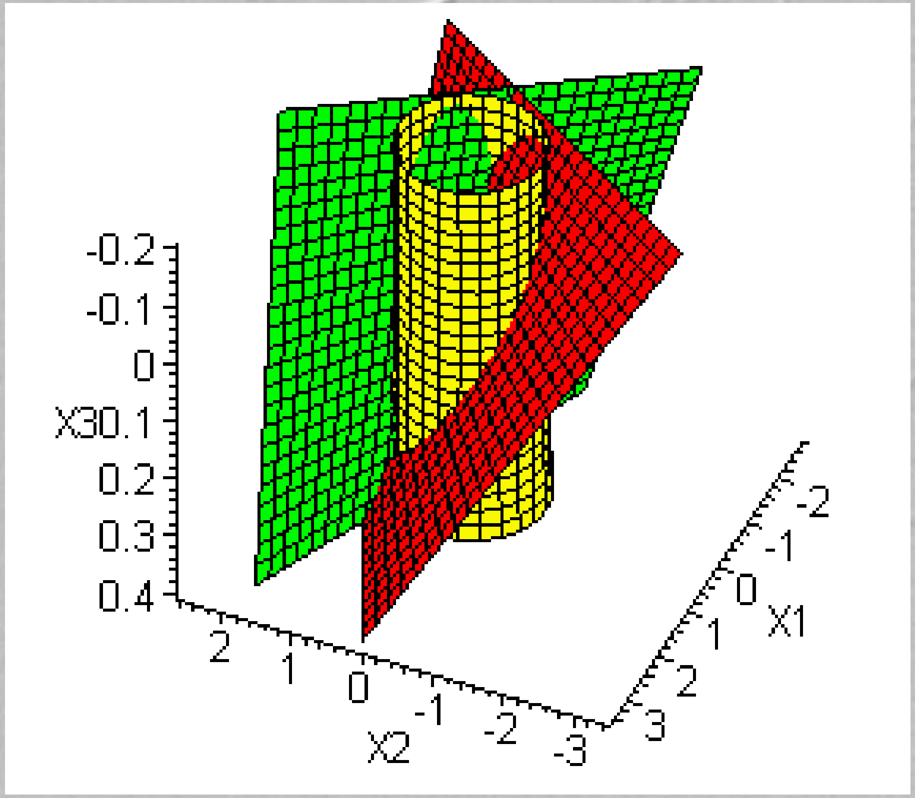
Kinematic Constraints

- *RR*-type legs: hyperboloid
  - One of the passive *R*-pairs has fixed position in Σ. The other, with fixed position in *E*, moves on a circle of fixed radius centred on the stationary *R*-pair.
- *PR*-type legs: hyperbolic paraboloid
  - The passive *R*-pair, with fixed position in *E*, is constrained to move on a line with fixed line coordinates in Σ.
- *RP*-type legs: hyperbolic paraboloid
  - The passive *P*-pair, with fixed position in *E*, is constrained to move on a point with fixed point coordinates in Σ. These are kinematic inversions, or projective duals, of the *PR*-type platforms.

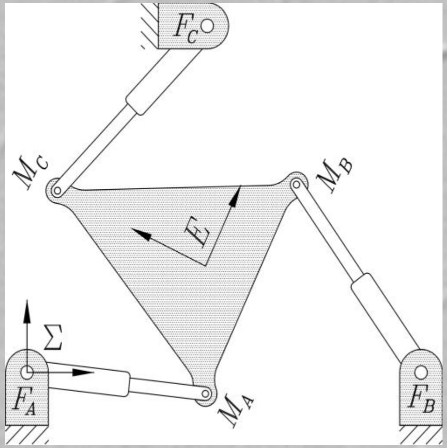


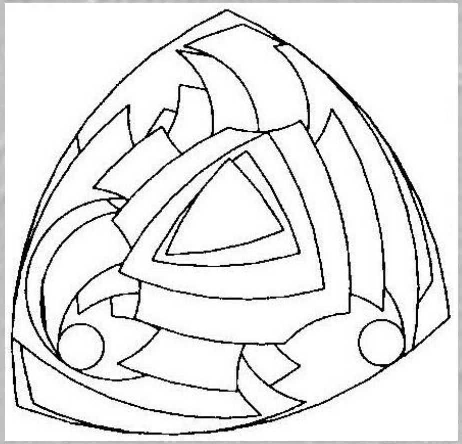
The Direct Kinematic Problem

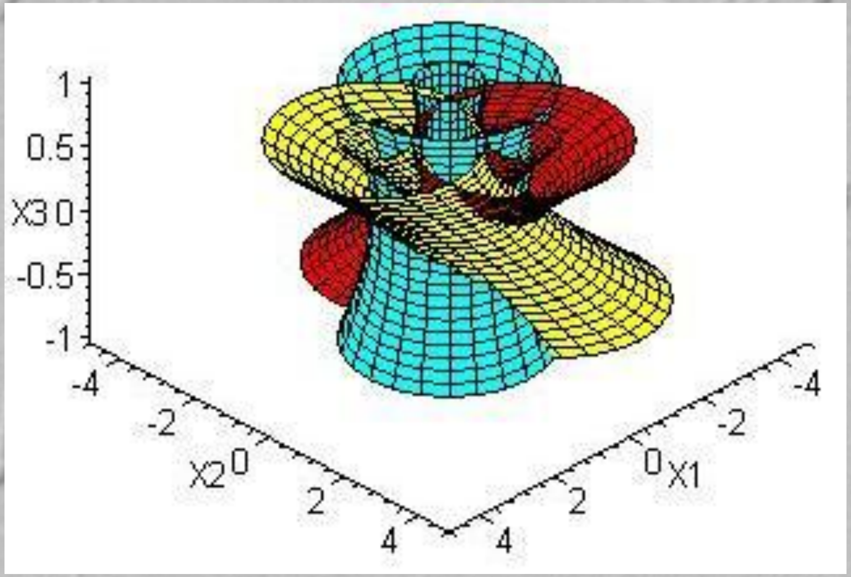
- The direct kinematic position analysis of any planar three-legged platform jointed with lower-pairs reduces to evaluating the points common to three quadric surfaces.

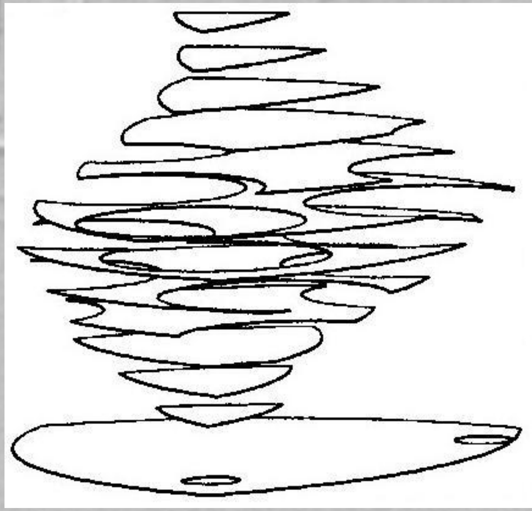


Workspace Visualization: Three *RPR*-Type Legs

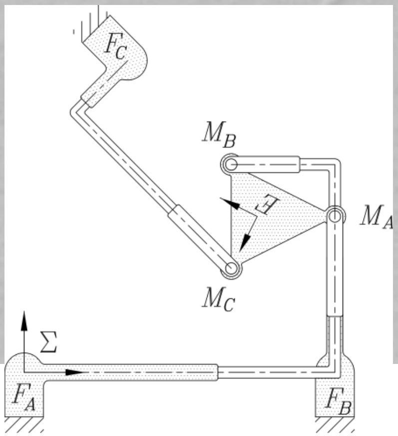


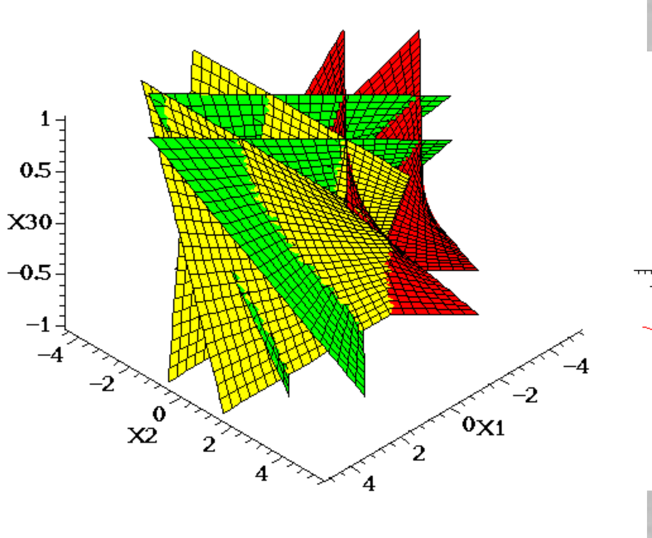


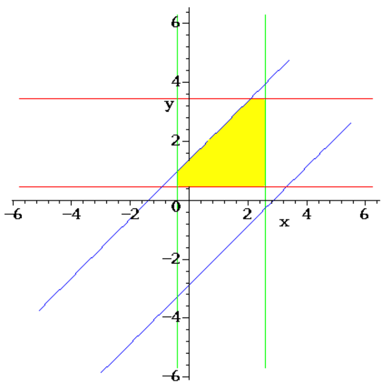




Three *PP̲R*-Type Legs







Cartesian layer $\phi = 150^o$

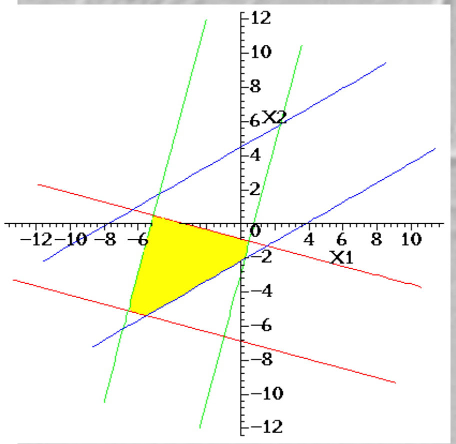
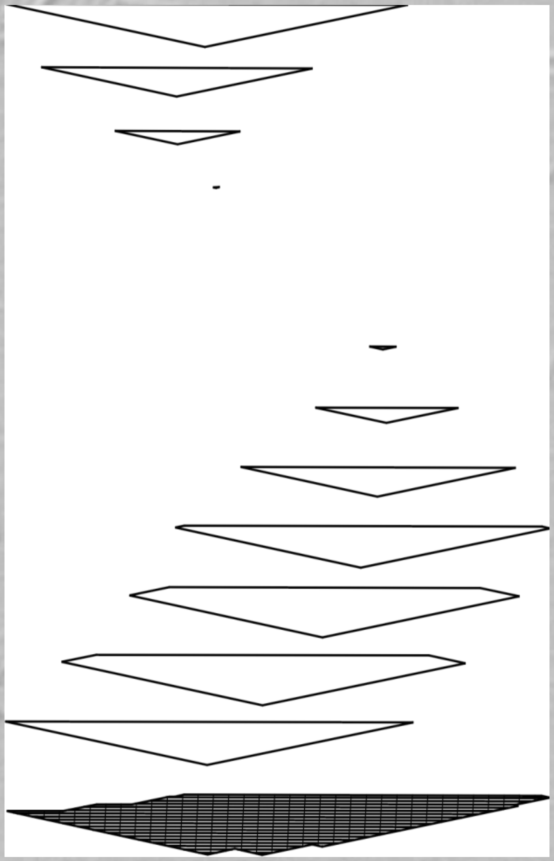
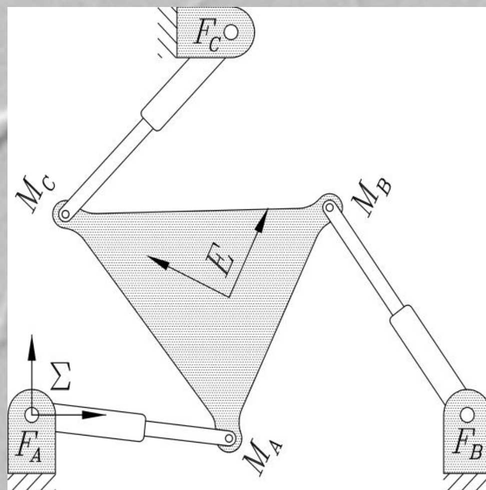
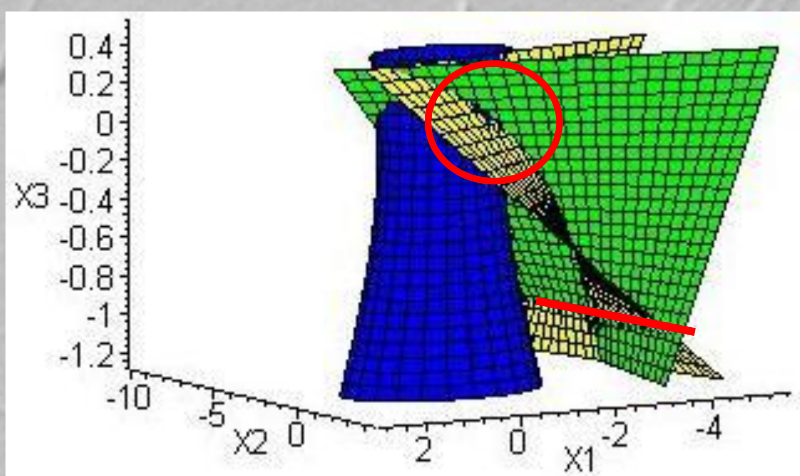


Image space layer $\phi = 150^o$



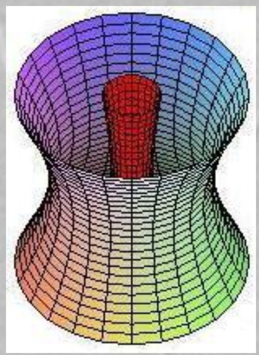
Mixed Leg ($R\underline{P}R$, $\underline{R}PR$, $RP\underline{R}$) Platform

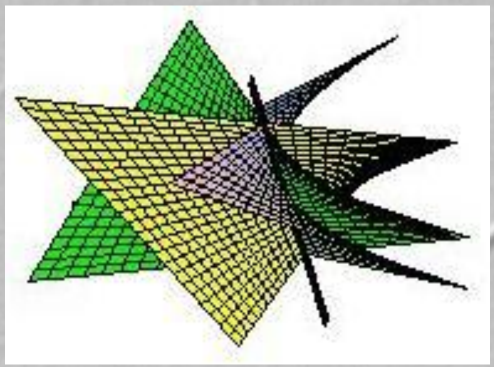


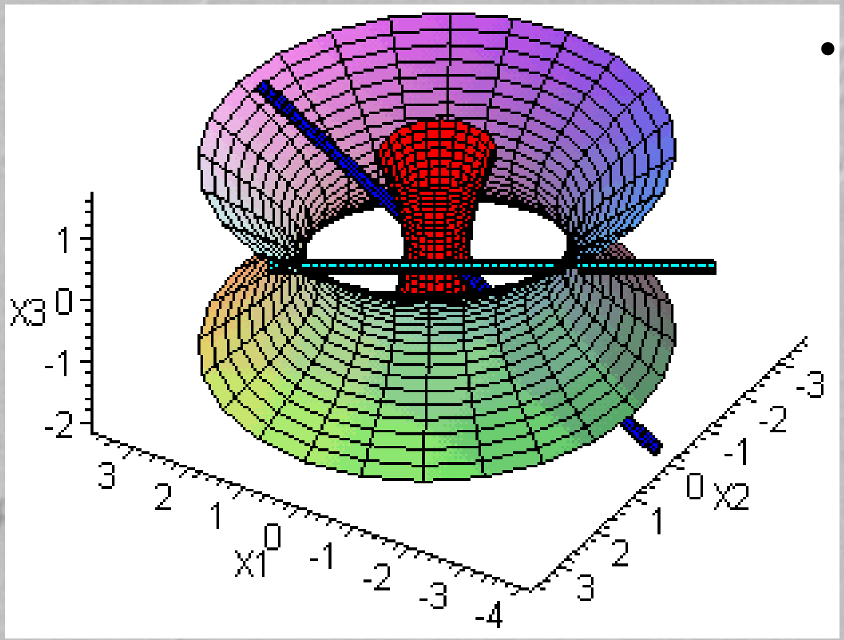


- This particular platform consists of one each of *RR*-type, *RP*-type and *PR*-type legs.
- The constraint surfaces for given leg inputs define the 3 constraint surfaces.
- The surfaces reveal 2 real and a pair of complex conjugate FK solutions.
- The $\underline{R}PR$ and $RP\underline{R}$ constraint surfaces have a common generator.

Mixed Leg Platform Workspace







- *RR*-type legs result in families of hyperboloids of one sheet all sharing the same axis.
- *PR*- and *RP*-type legs in general result in families of hyperbolic paraboloids.
- These families are pencils:
  - If the active joint is a *P*-pair the hyperbolic paraboloids in one family share a generator on the plane at infinity.
  - If the active joint is an *R*-pair the hyperbolic paraboloids in one family share a finite generator.